MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

O. M. BEKETOV NATIONAL UNIVERSITY
of URBAN ECONOMY in KHARKIV

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HIGHER MATHEMATICS

Module 1

LECTURE NOTES

(for full-time students (bachelor) of the education level of the specialty 122 – Computer science)

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The lectures notes are compiled to help students of computer science specialties of university in preparation for classes and exams in higher mathematics.

CONTENT

PREFACE

The purpose of the discipline is to provide a proper fundamental mathematical training of students and to form in them knowledge and ability to apply it for the analysis of various phenomena according to diversity spheres of a professional activity. Thus, the task of the discipline is to assist students learn the basics of mathematical apparatus needed to solve theoretical and practical problems, to develop skills and abilities of mathematical research of applied tasks, to develop their analytical and critical thinking, to teach students to understand the scientific sources of professional applications of mathematics.

This syllabus of lectures is designed according to the program of normative educational discipline "Higher mathematics" and the working curriculum of preparation of full-time first year students of the "bachelor" education level of the specialty 122 – Computer science.

All theoretical material in this lecture notes is structured and coordinated with the classroom lectures conducted during the study of Module 1 topics.

However, this synopsis is not final, because the volume of the studied theoretical material may be changed due to some changes in the curriculum. Therefore, students should follow the classroom lectures carefully and use a wider range of scientific and literary sources, which are presented at the end of the lecture notes, in their preparation for all class.

The lecture notes contain theoretical material as a basic knowledge of the topics of Module 1 that students need to acquire, and self-checking questions.

The lecture notes have a significant number of examples of solving typical tasks, as well as applied tasks, that help student to switch their attention to the practical using of the knowledge to solve professional-oriented tasks.

Some additional information and interest materials are in the appendices, at the end of the lecture notes.

So many references to sources in which students can find more detailed information about certain mathematical positions or theorems proofs that are not presented in this lecture notes are also given here as an aid to a more in-depth study and search for reference information.

The presented lecture notes will help students to possess the methods of solving practical tasks; it will promote the acquisition of mathematical competencies and intensify students' independent work.

Students must realize that only active work with lecture notes can help them to be successful in the study of higher mathematics, achieve professional excellence.

CONTENTS OF MODULE 1

Module 1 Linear and vector algebra. Introduction to mathematical analysis. Integral calculus

Content module 1.1 Linear algebra and analytic geometry

Matrices and actions on them. Determinants and their properties. Systems of linear algebraic equations. Solving quadratic systems using an inverse matrix, by Cramer's formulas. Rouch´e-Capelli theorem. Solving systems by the Gaussian elimination method. Vectors and actions on them*. Straight line on the plane*. Second order curves*. Polar coordinate system*. Parametrically form lines*. Straight line and plane in the space*. Second order surfaces*

Content module 1.2 Complex numbers, elementary functions, functions of several variables

Complex numbers and actions on them. Vector and complex functions of a real variable. The concept of the function of a complex variable. Limits theory. The first and second standard limits. Uncertainties and their disclosure. Function. Elementary functions. Continuity*. Properties of continuous functions*. Functions of many variables*. Area of definition*. Level lines and surfaces*. Limit and continuity of the function of many variables*

Content module 1.3 Differential calculus, integral calculus

Derivative and its properties. Derivatives of higher orders. Function differential and its properties. Basic theorems of differential calculus. Conditions of the function decreasing and increasing. Necessary and sufficient conditions for the function extremum. The smallest and largest value of the function on the segment. Conditions of convexity and concavity of the function graph*. Antiderivative function and indefinite integral. Integration methods. Defined integral. Newton and Leibniz formula. Improper integrals*. Geometric applications of a definite integral

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* self-study topics

Lecture 1 **LINEAR ALGEBRA. MATRICES and DETERMINANTS**

Definition 1.1 A matrix is defined as an ordered rectangular array of numbers. They can be used to represent systems of linear equations, as will be explained further.

You can see some examples of different types of matrices:

Symmetric	Diagonal	Upper Triangular	Lower Triangular	Zero	Identity
$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -5 \\ 3 & -5 & 6 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 7 & -5 \\ 0 & 0 & -4 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 7 & 0 \\ 12 & 5 & 3 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

On the right is an example of a 2×4 matrix. It has 2 rows and 4 columns. We usually write matrices inside parentheses ( ) or brackets [ ]. We can add, subtract, and multiply matrices together, under certain conditions.

We use matrices to solve problems in electronics, statics, robotics, linear programming, optimization, intersections of planes, genetics.

We will use matrices to solve systems of equations, but for large systems of equations, it is advisable to use a computer to find the solution. However, we should understand what the computer is doing for it and have opportunity to correct mistakes if it needs.

And a fully expanded $m \times n$ matrix A, would look like this:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

or in a more compact form: $A = (a_{ij})$

Matrix addition and subtraction

Definition 1.2 Two matrices A and B can be added or subtracted if and only if their dimensions are the same (i.e., both matrices have the same number of rows and columns). Take:

$$A = \begin{pmatrix} 3 & -4 & 2 \\ -8 & 1 & 5 \end{pmatrix},\ B = \begin{pmatrix} -7 & 5 & -4 \\ -8 & 3 & 2 \end{pmatrix}.$$

Addition

If A and B above are matrices of the same type, then the sum is found by adding the corresponding elements $a_{ij} + b_{ij}$. For example,

$$A + B = \begin{pmatrix} 3 & -4 & 2 \\ -8 & 1 & 5 \end{pmatrix} + \begin{pmatrix} -7 & 5 & -4 \\ -8 & 3 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 3-7 & -4+5 & 2-4 \\ -8-8 & 1+3 & 5+2 \end{pmatrix} = \begin{pmatrix} -4 & 1 & -2 \\ -16 & 4 & 7 \end{pmatrix}$$

Subtraction

If A and B are matrices of the same type, then the subtraction is found by subtracting the corresponding elements $a_{ij} - b_{ij}$. Here is an example of subtracting matrices:

$$A - B = \begin{pmatrix} 3 & -4 & 2 \\ -8 & 1 & 5 \end{pmatrix} - \begin{pmatrix} -7 & 5 & -4 \\ -8 & 3 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 3+7 & -4-5 & 2+4 \\ -8+8 & 1-3 & 5-2 \end{pmatrix} = \begin{pmatrix} 10 & -9 & 6 \\ 0 & -2 & 3 \end{pmatrix}$$

Matrix multiplication

Definition 1.3 When the number of columns of the first matrix is the same as the number of rows in the second matrix then matrix multiplication can be performed.

Here is an example of matrix multiplication for two 2×2 matrices.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} (ae+bg) & (af+bh) \\ (ce+dg) & (cf+dh) \end{pmatrix}$$

Here is an example of matrix multiplication for two 3×3 matrices:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \cdot \begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix} =$$

$$= \begin{pmatrix} (aj+bm+cp) & (ak+bn+cq) & (al+bo+cr) \\ (dj+em+fp) & (dk+en+fq) & (dl+eo+fr) \\ (gj+hm+ip) & (gk+hn+iq) & (gl+ho+ir) \end{pmatrix}$$

Now let's look at the multiplication of the arbitrary matrices, in the case, where A has dimensions $m \times n$, B has dimensions $n \times p$. Then the product of A and B is the matrix C, which has dimensions $m \times p$. The ij^{th} element of matrix C is found by multiplying the entries of the i^{th} row of A with the corresponding entries in the j^{th} column of B and summing the n terms.

In conclusion we should write down some remarks.

Remark 1.1. That $A \times B$ is not the same as $B \times A$.

Remark 1.2. Multiplication matrices can be performed only if the number of columns in the first matrix is equal to the number of rows of the second matrix:

$$A_{m\times k} \cdot B_{k\times n} = D_{m\times n}.$$

For example, take $A=\begin{pmatrix}3 & -1 & 1\\ 2 & -2 & 3\end{pmatrix}$ and $B=\begin{pmatrix}7 & 5\\ 0 & -8\end{pmatrix}$, find $A\cdot B$ and $B\cdot A$.

Solution: 1) $A\cdot B=\begin{pmatrix}3 & -1 & 1\\ 2 & -2 & 3\end{pmatrix}\cdot\begin{pmatrix}7 & 5\\ 0 & -8\end{pmatrix}$, as we see, we can't multiply these matrices, because the number of columns in the first matrix are 3, and the number of rows in the second matrix are 2; but multiplication $B\times A$ exists, find it:

2) We multiply the individual elements along the first row of matrix A with the corresponding elements down the first column of matrix *B*, and add the results and continue to do it with all rows

$$B\cdot A=\begin{pmatrix}7 & 5\\ 0 & -8\end{pmatrix}\cdot\begin{pmatrix}3 & -1 & 1\\ 2 & -2 & 3\end{pmatrix}=$$

$$\begin{pmatrix}7\cdot 3+5\cdot 2 & -1\cdot 7+5\cdot(-2) & 7\cdot 1+5\cdot 3\\ 0\cdot 3+(-8)\cdot 2 & 0\cdot(-1)-8\cdot(-2) & 0\cdot 1+(-8)\cdot 3\end{pmatrix}=\begin{pmatrix}31 & -17 & 22\\ -16 & 16 & -24\end{pmatrix}.$$

We need to know about the transposed matrix.

Definition 1.4 The transpose of a matrix is found by exchanging rows for columns Matrix $A=(a_{ij})$ and the transpose of A^T is: $A^T=(a_{ji})$, where j is the column number and i is the row number of matrix A. *For example*, the transpose of a matrix would be:

$$A=\begin{pmatrix}-7 & 2 & -3\\ 3 & -5 & 2\end{pmatrix};\quad A^T=\begin{pmatrix}-7 & 3\\ 2 & -5\\ -3 & 2\end{pmatrix}$$

In the case of a square matrix ($m=n$), the transpose can be used to check if a matrix is symmetric. For a symmetric matrix: $A=A^T$.

$$A=\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix};\quad A^T=\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix}$$

The determinant of a matrix

Definition 1.5 Determinants play an important role in finding the inverse matrix and in a solving system of linear equations. In the following we assume that we have a square matrix ($m = n$). The determinant of a matrix A will be denoted by $\det A$ or $|A|$. Firstly, the determinant of a 2×2 and 3×3 matrix will be introduced, then the $n\times n$ case will be shown.

Determinant of a 2×2 matrix

Assuming A is an arbitrary 2×2 matrix A, where the elements are given by:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

then the determinant of this matrix is as follows:

$$\det A = |A| = \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\cdot a_{22} - a_{21}\cdot a_{12}.$$

For example, calculate the given determinant $\begin{vmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{vmatrix}$.

Using the appropriate rule and we get

$$\begin{vmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{vmatrix} = \cos 2x\cdot\cos 2x - \sin 2x\cdot(-\sin 2x) =$$

$$= \cos^2 2x + \sin^2 2x = 1.$$

Determinant of a 3×3 matrix

The determinant of a 3×3 matrix is a little more tricky and is found as follows (for this case assume A is an arbitrary 3×3 matrix A, where the elements are given below).

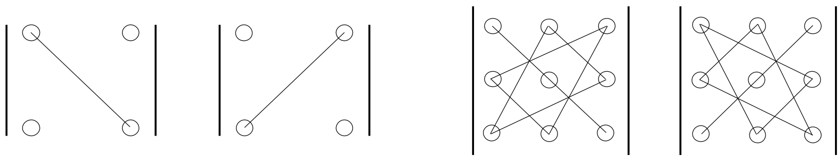
$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

then the determinant of a this matrix is as follows:

$$\Delta_3=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}\cdot a_{22}\cdot a_{33}+a_{12}\cdot a_{23}\cdot a_{31}+$$

$$+a_{21}\cdot a_{32}\cdot a_{13}-a_{13}\cdot a_{22}\cdot a_{31}-a_{12}\cdot a_{21}\cdot a_{33}-a_{32}\cdot a_{23}\cdot a_{11}$$

This rule can be represented schematically as:



"+" "-" "+" "-"

For example, calculate the determinant $\begin{vmatrix} 2 & 3 & -2 \\ -1 & -2 & 3 \\ 5 & 4 & -2 \end{vmatrix}$.

Solution. Using the previous reviewed rule

$$\Delta_3=\begin{vmatrix} 2 & 3 & -2 \\ -1 & -2 & 3 \\ 5 & 4 & -2 \end{vmatrix}=2\cdot(-2)\cdot(-2)+3\cdot 3\cdot 5+$$

$$(-1)\cdot 4\cdot(-2)-5\cdot(-2)\cdot(-2)-3\cdot(-1)\cdot(-2)-3\cdot 4\cdot 2=$$

$$=-8+45+8-20-6-24=-5.$$

Determinant of an $n \times n$ matrix

For the general case, where A is an $n \times n$ matrix, its determinant is given by the rule: the determinant is equal to the sum of items of elements of a certain row (column) on their cofactor

$$\boxed{\Delta = \sum_{k=1}^{n} a_{ik} \cdot A_{ik} \text{ or } \Delta = \sum_{k=1}^{n} a_{kj} \cdot A_{kj}},$$

where $\boxed{A_{ij} = (-1)^{i+j} \cdot M_{ij}}$ is a cofactor of element of a_{ij}

Also, M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix that is obtained by deleting row *i* and column *j*.

For example, we will calculate the determinant

$$\begin{vmatrix} 2 & 1 & 5 & 1 \\ 3 & 2 & 1 & 2 \\ 1 & 2 & 3 & -4 \\ 1 & 1 & 5 & 1 \end{vmatrix} = 2 \cdot A_{11} + 3 \cdot A_{21} + 1 \cdot A_{31} + 1 \cdot A_{41} =$$

$$= 2 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & -4 \\ 1 & 5 & 1 \end{vmatrix} + 3 \cdot (-1)^{2+1} \cdot \begin{vmatrix} 1 & 5 & 1 \\ 2 & 3 & -4 \\ 1 & 5 & 1 \end{vmatrix} +$$

$$+ 1 \cdot (-1)^{3+1} \cdot \begin{vmatrix} 1 & 5 & 1 \\ 2 & 1 & 2 \\ 1 & 5 & 1 \end{vmatrix} + 1 \cdot (-1)^{4+1} \cdot \begin{vmatrix} 1 & 5 & 1 \\ 2 & 1 & 2 \\ 2 & 3 & -4 \end{vmatrix} =$$

$$2 \cdot (6 + 20 - 4 - 6 - 2 + 40) + 3 \cdot (-1) \cdot 0 + 1 \cdot 0 +$$

$$+ 1 \cdot (-4 + 20 + 6 - 2 - 6 + 40) = 108 + 54 = 162 .$$

The Inverse of a Matrix

Definition1.6 Assuming we have a square matrix A, which is non-singular (i.e. $\det A$ does not equal zero), then there exists an $n \times n$ matrix A^{-1} which is called the inverse of A, such that this property holds:

$$\boxed{A \cdot A^{-1} = A^{-1} \cdot A = E},$$

where E is the **identity matrix**.

We will find the inverse matrix of matrix $A = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$.

Solution. Calculate the determinant of matrix A

$\det A = \begin{vmatrix} -1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 2 \neq 0$, thus matrix A is non-singular and the inverse of A is. Find the transposed matrix A^T, it was $A^T = \begin{pmatrix} -1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$. We will find all the cofactor of element the transposed matrix A^T and write down the inverse matrix A^{-1}:

$$A_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = -1, \qquad A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1,$$

$$A_{13} = (-1)^{3+1} \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 3, \qquad A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0,$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}-1 & 1\\ 1 & 1\end{vmatrix}=-2, \qquad A_{23}=(-1)^{2+3}\begin{vmatrix}-1 & 2\\ 1 & 2\end{vmatrix}=4,$$

$$A_{31}=(-1)^{1+3}\begin{vmatrix}2 & 1\\ -1 & 0\end{vmatrix}=1, \qquad A_{32}=(-1)^{2+3}\begin{vmatrix}-1 & 1\\ 1 & 0\end{vmatrix}=1,$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}-1 & 2\\ 1 & -1\end{vmatrix}=-1, \qquad A^{-1}=\frac{1}{2}\begin{pmatrix}-1 & -1 & 3\\ 0 & -2 & 4\\ 1 & 1 & -1\end{pmatrix}.$$

Check out:

$$A^{-1}\cdot A=\frac{1}{2}\begin{pmatrix}-1 & -1 & 3\\ 0 & -2 & 4\\ 1 & 1 & -1\end{pmatrix}\cdot\begin{pmatrix}-1 & 1 & 1\\ 2 & -1 & 2\\ 1 & 0 & 1\end{pmatrix}=$$

$$=\frac{1}{2}\begin{pmatrix}-1\cdot(-1)-1\cdot 2+3\cdot 1 & -1\cdot 1-1\cdot(-1)+3\cdot 0 & -1\cdot 1-1\cdot 2+3\cdot 1\\ 0-2\cdot 2+4\cdot 1 & 0-2\cdot(-1)+4\cdot 0 & 0-2\cdot 2+4\cdot 1\\ 1\cdot(-1)+1\cdot 2-1\cdot 1 & 1\cdot 1+1\cdot(-1)+0 & 1\cdot 1+1\cdot 2-1\cdot 1\end{pmatrix}=$$

$$=\frac{1}{2}\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=E \quad .$$

Lecture 2 **SOLVING SYSTEMS OF EQUATIONS USING MATRICES AND DETERMINANTS**

Definition 2.1 A system of linear equations is a set of equations with m equations and n variables, is of the form of

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m . \end{cases}$$

Variables are denoted by $x_1, x_2, \ldots, x_n$ and the coefficients (a and b above) are assumed to be given. In matrix form the system of equations above can be written as:

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{pmatrix}$$

A simplified way of writing above is like this:

$$\boxed{A \cdot X = B},$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_m \end{pmatrix}; \quad A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}; \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{pmatrix}.$$

After looking at this we will now look at two methods used to solve matrices. These are: Inverse Matrix Method, Cramer's Rule.

Inverse Matrix Method

Definition 2.2 The inverse matrix method uses the inverse of a matrix to help solve a system of equations, such like the above $A \cdot X = B$. By pre-multiplying both sides of this equation by A^{-1} gives:

$A^{-1} \cdot AX = A^{-1} \cdot B$, as we know that $A A^{-1} = A^{-1} A = E$,

and we get

$$E \cdot X = A^{-1} \cdot B$$

or alternatively

$$\boxed{X = A^{-1} \cdot B}.$$

So, by calculating the inverse of the matrix and multiplying this by the matrix-column B we can find the solution to the system of equations directly.

From the above it is clear that the existence of a solution depends on the value of the determinant of A. There are three cases:

1) if the $\det A$ does not equal zero then solutions exist using;
2) if the $\det A$ is zero and $B = 0$ then the solution will not be unique or does not exist;
3) if the $\det A$ is zero and $B = 0$ then the solution can be $X = 0$ but as with 2 is not unique or does not exist.

Looking at system of equations we might have this

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = d_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = d_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = d_2 \end{cases}.$$

Written in matrix form would look like

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

and by rearranging we would get that the solution would look like

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} \cdot \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix},$$

where $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} = A^{-1}$ is the inverse matrix and we have to find it due to following steps:

1. We should find the determinant of the matrix A, it will be $\det A$.
2. We should transpose matrix A and obtain the matrix A^T
3. We should find all cofactors of each element of the matrix A^T and compose them in matrix A^*
4. The obtained matrix A^* should be multiply by number $\frac{1}{\det A}$ (the inverse of the value of the determinant), so it will be the desired matrix A^{-1}.

After finding the inverse matrix, it is multiplied by the matrix B and we will find the matrix-column, this is the required value of variables x_1, x_2, x_3. At the end we have to check our solution to substitute the obtained values in equations of the system.

For example, solve a system of linear algebraic equations by the matrix method

$$\begin{cases} 3x_1 - 2x_2 + x_3 = 1 \\ x_1 - x_2 + 2x_3 = -3 \\ 2x_1 - x_2 + 3x_3 = -4 \end{cases}.$$

Solution. Find the inverse matrix of the system

$$A = \begin{pmatrix} 3 & -2 & 1 \\ 1 & -1 & 2 \\ 2 & -1 & 3 \end{pmatrix}, \ \det A = \begin{vmatrix} 3 & -2 & 1 \\ 1 & -1 & 2 \\ 2 & -1 & 3 \end{vmatrix} = -4 \neq 0 .$$

$$A^T = \begin{pmatrix} 3 & 1 & 2 \\ -2 & -1 & -1 \\ 1 & 2 & 3 \end{pmatrix}, \ A_{11} = -1, \ A_{12} = 5, \ A_{13} = -3, \ A_{21} = 1,$$

$$A_{22} = 7, \ A_{23} = -5, \ A_{31} = 1, \ A_{32} = -1, \ A_{33} = -1 .$$

$$A^* = \begin{pmatrix} -1 & 5 & -3 \\ 1 & 7 & -5 \\ 1 & -1 & -1 \end{pmatrix}, \ A^{-1} = \frac{1}{-4}\begin{pmatrix} -1 & 5 & -3 \\ 1 & 7 & -5 \\ 1 & -1 & -1 \end{pmatrix}.$$

$$X = A^{-1} \cdot B = \frac{1}{-4}\begin{pmatrix} -1 & 5 & -3 \\ 1 & 7 & -5 \\ 1 & -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} -4 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix},$$

$$x_1 = 1, \ x_2 = 0, \ x_3 = -2 .$$

After solution we need to check the found variable value, we substitute the values of variables x_1, x_2, x_3 in the second equation and obtain the identity: $1 - 0 + 2 \cdot (-2) = -3$. So, it is correct answer.

Cramer's Rule

Definition 2.3 Cramer's rule uses a method of determinants to solve systems of equations. Starting with equation below,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = d_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = d_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = d_2 \end{cases}$$

The first we should calculate the main determinant of the system is composed of the coefficients for variables as a

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

After this we should calculate **auxiliary determinants** each of which is obtained by successively replacing the columns of the determinant by a column of numbers that are following in the equations of the system after the sign of equal. Doing this we obtain three determinants:

$$\Delta_1 = \begin{vmatrix} d_1 & a_{12} & a_{13} \\ d_2 & a_{22} & a_{23} \\ d_3 & a_{32} & a_{33} \end{vmatrix}, \ \Delta_3 = \begin{vmatrix} a_{11} & d_1 & a_{13} \\ a_{21} & d_2 & a_{23} \\ a_{31} & d_3 & a_{33} \end{vmatrix}, \ \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & d_1 \\ a_{21} & a_{22} & d_2 \\ a_{31} & a_{32} & d_3 \end{vmatrix}.$$

And we can find values of variables x_1, x_2, x_3 by formulas

$$x_1 = \frac{\Delta_1}{\det A}, \ x_2 = \frac{\Delta_2}{\det A}, \ x_3 = \frac{\Delta_3}{\det A}.$$

Example 2.1 Solve a system of linear algebraic equations by Cramer's rule

$$\begin{cases} 5x_1 - 3x_2 + x_3 = 2 \\ 3x_1 + x_2 - 5x_3 = 4 \\ x_1 - 2x_2 + 7x_3 = 3 \end{cases}$$

Solution. The main determinant and auxiliary determinants are being calculating as

$$\Delta = \begin{vmatrix} 5 & -3 & 1 \\ 3 & 1 & -5 \\ 1 & -2 & 7 \end{vmatrix} = 56 \neq 0, \quad \Delta_1 = \begin{vmatrix} 2 & -3 & 1 \\ 4 & 1 & -5 \\ 3 & -2 & 7 \end{vmatrix} = 112,$$

$$\Delta_2 = \begin{vmatrix} 5 & 2 & 1 \\ 3 & 4 & -5 \\ 1 & 3 & 7 \end{vmatrix} = 168, \quad \Delta_3 = \begin{vmatrix} 5 & -3 & 2 \\ 3 & 1 & 4 \\ 1 & -2 & 3 \end{vmatrix} = 56 \cdot$$

And values of x_1, x_2, x_3 are finding: $x_1 = \dfrac{\Delta_1}{\Delta} = \dfrac{112}{56} = 2$,

$$x_2 = \frac{\Delta_2}{\Delta} = \frac{168}{56} = 3, \quad x_3 = \frac{\Delta_3}{\Delta} = \frac{56}{56} = 1.$$

Check out: $5 \cdot 2 - 3 \cdot 3 + 1 = 2$, $2 = 2$.

Answer: $x_1 = 2$, $x_2 = 3$, $x_3 = 1$.

Systems of linear equations: solving by Gaussian elimination

In linear algebra, **Gaussian elimination** (also known as **row reduction**) is an algorithm for solving systems of linear equations. It is usually understood as a sequence of operations performed on the corresponding matrix of coefficients. This method can also be used to find the rank of a matrix, to calculate the determinant of a matrix, and to calculate the inverse of an invertible square matrix. The method is named after Carl Friedrich Gauss (1777–1855), although it was known to Chinese mathematicians as early as 179 CE.

To perform row reduction on a matrix, one uses a sequence of elementary row operations to modify the matrix until the lower left-hand corner of the matrix is filled with zeros, as much as possible. There are three types of elementary row operations:

1) swapping two rows;

2) multiplying a row by a non-zero number;

3) adding a multiple of one row to another row.

Using these operations, a matrix can always be transformed into an upper triangular matrix, and in fact that is named as a staged matrix form. Once all of the coefficients under the main diagonal is 0, the matrix is said to be in reduced row echelon form (a staged matrix form). After that we are writing down a new system of equations and from this system, we are finding the value of our unknowns. This final form is unique; in other words, it is independent of the sequence of row operations used. For example, in the following sequence of row operations (where multiple elementary operations might be done at each step), the next obtained matrices are the ones in row echelon form, and the final matrix is the unique reduced row echelon form.

Example 2.2 Solve the following system of equations using Gaussian elimination method:

$$\begin{cases} 5x_1 - 3x_2 + x_3 = 2 \\ 3x_1 + x_2 - 5x_3 = 4 \\ x_1 - 2x_2 + 7x_3 = 3 \end{cases}$$

Solution: We write down the matrix with number of the system, as

$$\begin{cases} 5x_1 - 3x_2 + x_3 = 2 \\ 3x_1 + x_2 - 5x_3 = 4 \\ x_1 - 2x_2 + 7x_3 = 3 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 5 & -3 & 1 & 2 \\ 3 & 1 & -5 & 4 \\ 1 & -2 & 7 & 3 \end{array}\right) \Rightarrow (R_1 \Leftrightarrow R_3) \Rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 7 & 3 \\ 3 & 1 & -5 & 4 \\ 5 & -3 & 1 & 2 \end{array}\right)$$

For convenience, we will swap the first row by the third row. We exclude the first coefficients from the first column, which are below the first row. To do this, we add the first row, multiplied by (-3), to the second row. Then we will exclude the coefficients from the second column, which are below the first row. Now we add the first row, multiplied by (-5), to the third. We perform actions in a consistent manner. As a result, we will have:

$$\left(\begin{array}{ccc|c} 5 & -3 & 1 & 2 \\ 3 & 1 & -5 & 4 \\ 1 & -2 & 7 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 7 & 3 \\ 3 & 1 & -5 & 4 \\ 5 & -3 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 7 & 3 \\ 0 & 1+(-2)\cdot(-3) & -5+7\cdot(-3) & 4+3\cdot(-3) \\ 5 & -3 & 1 & 2 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 7 & 3 \\ 0 & 7 & -26 & -5 \\ 0 & -3+(-2)\cdot(-5) & 1+7\cdot(-5) & 2+3\cdot(-5) \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 7 & 3 \\ 0 & 7 & -26 & -5 \\ 0 & 7 & -34 & -13 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 7 & 3 \\ 0 & 7 & -26 & -5 \\ 0 & -7+7 & 26-34 & -13+5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 7 & 3 \\ 0 & 7 & -26 & -5 \\ 0 & 0 & -8 & -8 \end{array}\right)$$

The resulting matrix has an upper triangular appearance; therefore, the system will have one singular solution, which we will find. Let's make and solve a system of equations

$$\begin{cases} x_1 - 2x_2 + 7x_3 = 3 \\ 7x_2 - 26x_3 = -5 \\ 8x_3 = 8 \end{cases} \Rightarrow \begin{cases} x_1 - 2x_2 + 7x_3 = 3 \\ 7x_2 - 26x_3 = -5 \\ x_3 = 1 \end{cases} \Rightarrow \begin{cases} x_1 - 2x_2 + 7x_3 = 3 \\ 7x_2 - 26\cdot 1 = -5 \\ x_3 = 1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_1 - 2x_2 + 7x_3 = 3 \\ 7x_2 = 21 \\ x_3 = 1 \end{cases} \Rightarrow \begin{cases} x_1 - 2\cdot 3 + 7 = 3 \\ x_2 = 3 \\ x_3 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 2 \\ x_2 = 3 \\ x_3 = 1 \end{cases}$$

Example 2.3 Solve the following system of equations using Gaussian elimination method

$$\begin{cases} x_1 - x_2 + 2x_3 - 3x_4 = 2 \\ 2x_1 + 2x_2 - 3x_3 + 4x_4 = -6. \\ 3x_1 + x_2 - x_3 + x_4 = 5 \end{cases}$$

Solution:
$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left(\begin{array}{cccc|c} 1 & -1 & 2 & -3 & 2 \\ 2 & 2 & -3 & 4 & -6 \\ 3 & 1 & -1 & 1 & 5 \end{array}\right) \end{array} \sim \left| \begin{array}{l} R_2 := R_2 - 2R_1 \\ R_3 := R_3 - 3R_1 \end{array} \right| \sim$$

$$\sim \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left(\begin{array}{cccc|c} 1 & -1 & 2 & -3 & 2 \\ 0 & 4 & -7 & 10 & -10 \\ 0 & 4 & -7 & 10 & -1 \end{array}\right) \end{array} \sim \left| \begin{array}{l} R_2 := R_2/4 \\ R_3 := R_3 - R_2 \end{array} \right| \sim$$

$$\sim \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left(\begin{array}{cccc|c} 1 & -1 & 2 & -3 & 2 \\ 0 & 1 & -7/4 & 5/2 & -5/2 \\ 0 & 0 & 0 & 0 & 9 \end{array}\right) \end{array}.$$

Since the last row corresponds to an equation with zero coefficients but a non-zero free term, the system is incompatible (no solutions).

Example 2.4 Solve a homogeneous system of linear algebraic equations by the Gaussian method:

$$\begin{cases} x_1 + 3x_2 + 2x_3 = 0; \\ 2x_1 - x_2 + 3x_3 = 0; \\ 3x_1 - 5x_2 + 4x_3 = 0. \end{cases}$$

Solution. We write the system in the form of an extended matrix (the ***augmented matrix***): $\overline{A}=\begin{pmatrix}1&3&2\\2&-1&3\\3&-5&4\end{pmatrix}$. Multiply the first row by (-2) and add it to the second row, then, multiply the first row by (-3) and add it to the third row, we get:

$$\begin{pmatrix}1&3&2\\0&-7&-1\\0&-14&-2\end{pmatrix}.$$

Multiply the second row by (-2) and add it to the third row:

$$\begin{pmatrix}1&3&2\\0&-7&-1\\0&0&0\end{pmatrix}.$$

After this we can write down a system in the following form:

$$\begin{cases}x_1+3x_2+2x_3=0;\\ \quad -7x_2-x_3=0;\\ \qquad\quad 0\cdot x_3=0.\end{cases}$$

From the third row we get that a variable x_3 can be an arbitrary value. Let it be: $x_3=t$. Substitute it at the second equation $-7x_2-t=0$, and get that $x_2=-\frac{t}{7}$. Substitute all of them at the first equation

$$x_1+3\cdot\left(-\frac{t}{7}\right)+2t=0,\quad x_1-\frac{3t}{7}+2t=0,\quad x_1=\frac{3t}{7}-2t,$$

$$x_1=\frac{3t-14t}{7},\quad x_1=\frac{-11t}{7}.$$

Answer: $x_1 = -\frac{11t}{7}$, $x_2 = -\frac{t}{7}$, $x_3 = t$, $t \in R$.

The matrix rank

We recall the definition of matrix minor.

Definition 2.4 Given the matrix $A = (a_{ij})_{m \times n}$, we call minor of order k the determinant of any square submatrix that can be constructed by A cutting (a certain number) of rows and/or columns.

From this definition it is clear, that the order of the minors that can be extracted by A cannot be larger than the minimum between m and n. Suppose indeed that $m \le n$ (i.e. there are less rows than columns), it is not possible to obtain a square matrix from A cutting rows or columns, whose dimension is larger than m.

Example 2.5 Consider $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 6 & 4 \end{pmatrix}$

Find all minors of the matrix A of order 2 and at least one minor of order 1 that is non-zero. Is it possible to obtain a minor of order $k > 2$?

Solution. The minors of order two are obtained by cutting one of the three columns of A. Hence, we can define three minors of order 2. These are

$$\begin{vmatrix} 2 & 1 \\ 1 & 6 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 6 & 4 \end{vmatrix}.$$

Recall that the notation $|A|$ states for the determinant of the matrix A. Hence

$$\begin{vmatrix} 2 & 1 \\ 1 & 6 \end{vmatrix} = 12 - 1 = 11, \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} = 8, \begin{vmatrix} 1 & 0 \\ 6 & 4 \end{vmatrix} = 4,$$

$$\begin{vmatrix} 1 & 1 \\ 6 & 4 \end{vmatrix} = 4 - 6 = -2 .$$

A minor of order one is the determinant of any entry of the matrix. For instance, $|a_{11}| = 2$ is a minor of A of order 1. The determinant of a scalar is the scalar itself so $|2| = 2$ which is different from zero. This is one example of non-zero minor of A of order 1. Concerning the existence of minors of order $k > 2$. Take for instance $k = 3$. It is not possible to find such a minor. In fact, cutting rows or columns of A, we cannot obtain a 3×3 matrix. Obviously, it is not possible to find an even larger one.

Now we state an important result that can be useful when computing the rank of a matrix.

Definition 2.5 We define $r(A)$ (rank of A) the maximum number of linearly independent rows or columns of A.

Remark 2.1 The rank of the matrix $A = \left(a_{ij}\right)_{m\times n}$ coincides with the order of the largest non-zero minor that can be extracted by A. Put differently we say that a matrix A has rank k if and only if there exists at least one minor different from zero of order k whereas all the minors of order larger than k are indeed zero. If there were a minor of order $k+1$ different from zero, then the rank of A would be at least $k+1$.

Return to the previous example and solve it in another way.

Answer. Since there exists at least one minor of A different from zero, for instance, $\begin{vmatrix} 2 & 1 \\ 1 & 6 \end{vmatrix} = 11,$, the rank of A is at least 2. Moreover, we have saw that it is not possible to extract a minor of order $k > 2$. Thus, the rank of A is 2.

Rules for the calculus of the rank

Given the matrix $A = \left(a_{ij}\right)_{m\times n}$:

1) $rank\ A$ is an integer number; 2) $rank\ A \geq 0$, in particular $rank\ A = 0$ if and only if A = 0, where 0 denotes the zero matrix; 3) $rank\ A \leq \min(m, n)$: the rank of A is at most equal to the minimum between the number of rows and columns; 4) as a consequence the following relationship holds: $0 \leq rank\ A \leq \min(m, n)$.

Rouch´e-Capelli Theorem. The system $AX = B$ admits solutions (it is consistent) if and only if $rank\ A = rank\ \overline{A}$ (where $\overline{A} = (A|B)$). Moreover, if the system is consistent, the number of degrees of freedom is equal to n it is a $rank\ A$, where n is the number of the system unknowns.

The first part of the theorem tells us whether there are solutions or not. The second part tells us that the solution is unique only if $rank\ A = n$. In this case in fact the number of the freedom degrees is zero. Otherwise, there are a positive number of the freedom degrees and thus there are infinite solutions.

Example 2.6 Solve the homogeneous system of linear algebraic equations

$$\begin{cases} x_1 - x_2 + 2x_3 = 0; \\ 2x_1 + x_2 - 3x_3 = 0; \\ 3x_1 + 2x_3 = 0. \end{cases}$$

Solution. Solve this system using the Gauss elimination method. Write down the expanded matrix (or the ***augmented matrix***) of the given system of equations:

$$\overline{A} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & -3 \\ 3 & 0 & 2 \end{pmatrix}.$$

Multiply the first row by (-2) and add it to the second row, multiply the first row by (-3) and add it to the third row, we get:

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -7 \\ 0 & 3 & -4 \end{pmatrix}$$

Multiply the second row by (-1) and add it to the third row

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -7 \\ 0 & 0 & 3 \end{pmatrix}.$$

Note, that the rank of the main matrix of a system is 2, and the rank of the expanded matrix (or the *augmented matrix*) is 2 too. Accordingly to the *Rouch′e-Capelli theorem* we make the conclusion that the given system has only one unique solution and we can find it.

Then we can write down a new system from the obtained matrix in a form as:

$$\begin{cases} x_1 - x_2 + 2x_3 = 0; \\ \quad 3x_2 - 7x_3 = 0; \\ \qquad\quad 3x_3 = 0. \end{cases}$$

We get from the third equation that, що $x_3 = 0$. From the second equation: $x_2 = 0$. From the first equation: $x_1 = 0$.

Check out:

$$\begin{cases} 0 - 0 + 2 \cdot 0 = 0; \\ 2 \cdot 0 + 0 - 3 \cdot 0 = 0; \\ 3 \cdot 0 + 2 \cdot 0 = 0. \end{cases}$$

Answer: $x_1 = x_2 = x_3 = 0$.

Lecture 3 ***COMPLEX NUMBERS. VECTOR AND COMPLEX FUNCTIONS OF A REAL VARIABLE***

For instance, $\sqrt{-9}$ isn't a real number since there is no real number that we can square and get a NEGATIVE 9.

Now we also saw that if a and b were both positive, then $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$. For a second let's forget that restriction and do the following.

$$\sqrt{-9} = \sqrt{-1 \cdot 9} = \sqrt{-1} \cdot \sqrt{9} = 3 \cdot \sqrt{-1}.$$

Now, $\sqrt{-1}$ is not a real number, but if you think about it, we can do this for any square root of a negative number. For instance,

$$\sqrt{-100} = \sqrt{-1}\sqrt{100} = 10 \cdot \sqrt{-1},$$

$$\sqrt{-81} = \sqrt{-1} \cdot \sqrt{81} = 9 \cdot \sqrt{-1} \text{ etc.}$$

So, even if the number isn't a perfect square, we can still always reduce the square root of a negative number down to the square root of a positive number (which we or a calculator can deal with) times $\sqrt{-1}$.

So, if we just had a way to deal with $\sqrt{-1}$ we could deal with square roots of negative numbers. Well, the reality is that, at this level, there just isn't any way to deal with $\sqrt{-1}$ so instead of dealing with it we will "make it go away" so to speak by using the following definition.

$$\sqrt{-1} = i.$$

Note that if we square both sides of this we get,

$$i^2 = -1.$$

It will be important to remember this later. This shows that, in some way, i is the only "number" that we can square and get a negative value.

Using this definition all the square roots above become,

$$\sqrt{-9}=3i,\ \sqrt{-81}=9i,\ \sqrt{-100}=10i.$$

These are all examples of *complex numbers*.

The natural question at this point is probably just why do we care about this? The answer is that, as we will see in the next lectures, sometimes we will run across the square roots of negative numbers and we're going to need a way to deal with them. So, to deal with them we will need to discuss complex numbers.

So, let's start out with some of the basic definitions and terminology for complex numbers. The standard form of a complex number is

$$z=x\pm iy,$$

where x and y are real numbers and they can be anything, positive, negative, zero, integers, fractions, decimals, it doesn't matter. When in the standard form x is called the *real part* of the complex number and y is called the *imaginary part* of the complex number. Denote it $x=\operatorname{Re} z;\quad y=\operatorname{Im} z$.

Here are some examples of complex numbers: $5-2i$, $7+4i$, $-8i$, 15.

The set (plural) of all complex numbers is denoted by C.

If the imaginary part is zero and we actually have a real number. So, thinking of numbers in this context we can see that the real numbers are simply a subset of the complex numbers. Any real number x can be presented as a complex number $z=x+i\,0=x$, in which the imaginary part is zero: $y=0$. Thus, the set of real numbers R is a subset of the set of complex numbers C: $R\subset C$. When the real part is zero, we often will call the complex number a *purely imaginary number:*

$$z=iy=0+iy,\ \ y\neq 0.$$

The conjugate of the complex number $z_1=x+iy$ is the complex number $z_2=x-iy$. In other words, it is the original complex number with the sign on the imaginary part changed. Here are some examples of complex numbers and their conjugates.

complex number	*conjugate*
$5+8i$	$5-8i$,
$-3-2i$	$-3+2i$.

Notice that the conjugate of a real number is just itself with no changes.

Definition 3.1 Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal, if their real and imaginary parts are equal, respectively:

$$z_1 = z_2 \Leftrightarrow x_1 = x_2 \;\; i \;\; y_1 = y_2 .$$

Now we need to discuss the basic operations for complex numbers. We'll start with addition and subtraction. The easiest way to think of adding and/or subtracting complex numbers is to think of each complex number as a polynomial and do the addition and subtraction in the same way that we add or subtract polynomials.

Definition 3.2 In particular, addition and subtraction of complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are carried out component by component:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2);$$
$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

Multiplication of the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are carried out by the rules of multiplication of the of binomials taking into account the condition $i^2 = -1$ and the reduction of similar:

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Remark 3.1 To multiply a complex number $z = x + iy$ by a real number a it is enough to multiply each of its components by this number a: $az = ax + iay$.

Remark 3.2 Find the natural powers of an imaginary unit:

$$i^2 = -1,\ i^3 = i^2 \cdot i = -i,\ i^4 = i^3 \cdot i = -i^2 = 1.$$

So,

$$i^{4k} = 1,\quad i^{4k+1} = i,\quad i^{4k+2} = -1,\quad i^{4k+3} = -i\ .$$

Remark 3.3 At raising of a complex number to a natural power it is possible to apply the formulas of the reduced multiplication known from elementary mathematics.

Remark 3.4 When we multiply or add a complex number by $z = x + iy$ and its conjugate $\bar{z} = x - iy$ we get a real number given by:

$$z + \bar{z} = 2x;\quad z \cdot \bar{z} = x^2 + y^2.$$

Definition 3.3 *Division of complex numbers* $z_1 = x_1 + iy_1$ i $z_2 = x_2 + iy_2$, $z_2 \neq 0$ is performed as follows: 1) the numerator and denominator of the fraction z_1/z_2 must be multiplied by the number $\bar{z}_2$ conjugated to the denominator z_2; 2) consider that $i^2 = -1$, and reduce similarities; 3) divide the numerator by the denominator and get the fraction in algebraic form.

$$z_1 : z_2 = \frac{z_1}{z_2} = \frac{z_1 \cdot \bar{z}_2}{z_2 \cdot \bar{z}_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.$$

Remark 3.5 The basic properties of the considered arithmetic operations with complex numbers coincide with the corresponding properties of similar operations with real numbers. Therefore, for complex numbers all theorems, rules, formulas derived for real numbers based on these properties remain valid.

Example 3.1 Do operations with complex numbers in the algebraic form:

$$z = (2-3i)(4+i) - (1-2i)^2 + 10\,(5-7i):(3-4i).$$

Solution. Do the operation with polynomials:

$$z = (2-3i)(4+i) - (1-2i)^2 + 10\,(5-7i):(3-4i) = (8+2i-$$

$$-12i - 3i^2 - 1 + 4i - 4i^2 + 10\big((5-7i)(3+4i)\big)/\big((3-4i)(3+4i)\big) =$$

$$= 8 + 2i - 12i + 3 - 1 + 4i + 4 + 10\left(15 + 20i - 21i - 28i^2\right):\left(9 -\right.$$

$$\left.-16i^2\right) = 14 - 6i + 10\left(15 + 20i - 21i + 28\right):(9+16) = 14 - 6i +$$

$$+2\left(43 - i\right):5 = (70 - 30i + 86 - 2i):5 = 156/5 - (32/5)\,i.$$

Geometric interpretation. Module and argument of the complex number

If we have a Cartesian coordinate plane Oxy, then it is possible a mutually unique correspondence between the set of all points of this plane and the set of complex numbers can be to established: each complex number $z = x + iy$ corresponds to a single point $M(x;y)$ and vice versa (fig. 3.1). Real numbers are represented by points on the abscissa Ox, therefore the axis Ox is called the real axis. Purely imaginary numbers are represented by points on the *y*-axis Oy, therefore the axis Oy is called the *imaginary axis*. Number $z = 0$ corresponds to the origin $O(0;0)$.

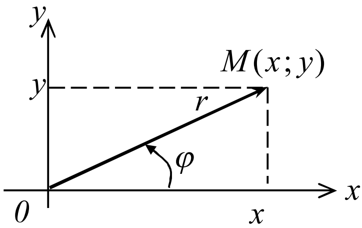


Figure 3.1

A coordinate plane Oxy, that represents the set of all complex numbers C, is called a *complex plane* C or z *- plane.*

Remark 3.6 The complex number $z = x + iy$ can also be represented by a radius vector $\overrightarrow{OM}(x;y)$, starting from the origin $O(0;0)$ and ending at a point $M(x;y)$ (fig. 3.1).

Remark 3.7 Addition and subtraction of complex numbers can be carried out according to the rules (triangle and parallelogram) of the corresponding operations on the vectors (fig. 3.2).

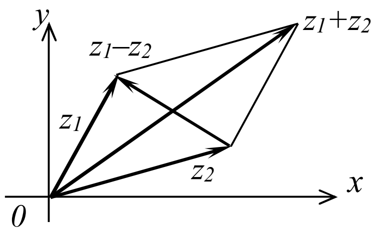


Figure 3.2

If (fig. 3.1) also enter a polar coordinate system $Or\varphi$ on the complex plane with a pole at the beginning of the Cartesian coordinate system and a polar axis aligned with the axis Ox, then the point $M(x;y)$ representing the complex number $z = x + iy$ can be set by polar coordinates $M(r;\varphi)$.

Definition 3.4 Polar radius r (length (magnitude) of a radius-vector $\overrightarrow{OM}$) is called a *module of a complex number* z and it is denoted as $|z| = r$.

Obviously, that $r = \sqrt{x^2 + y^2} \geq 0$.

Definition 3.5 Polar angel φ (the angel between the radius-vector $\overrightarrow{OM}$ and polar axis Ox) is called *an argument of a complex number* z and it is denoted as $Arg\, z = \varphi$.

Argument φ, is an angel of rotation, is determined with accuracy of the constant addition in the form $2\pi k$, $k = 0, \pm 1, \pm 2, \ldots$ (arbitrary number of full revolutions).

The single value φ that satisfies the condition $-\pi < \varphi \leq \pi$ is called the main value of the argument, and it is denoted, $\arg z$. So, $Arg\, z = \arg z + 2\pi k$, $k = 0, \pm 1, \pm 2, \ldots$

The main value of the argument is determined by the formula:

$$\arg z = \begin{cases} \operatorname{arctg}(y/x), & x > 0; \\ \operatorname{arctg}(y/x) + \pi, & x < 0, y \geq 0; \\ \operatorname{arctg}(y/x) - \pi, & x < 0, y < 0; \\ \pi/2, & x = 0; y > 0; \\ -\pi/2, & x = 0, y < 0. \end{cases}$$

Remark 3.8 The module of the number $z = 0$ is equal to zero $r = |0| = 0$, and the argument φ is arbitrary.

Remark 3.9 The equal complex numbers $z_1 = z_2$ have equal modules too, $r_1 = r_2$, and their arguments are related by a relation $\varphi_1 = \varphi_2 + 2\pi k$, $k = 0, \pm 1, \pm 2, \ldots$, that is, they are differ in addition $2\pi k$.

Trigonometric and exponential forms of a complex number

Using the relation of Cartesian and polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, the complex number $z = x + iy$ can be presented in a form

$$z = x + iy = r\cos\varphi + ir\sin\varphi = r(\cos\varphi + i\sin\varphi).$$

Definition 3.6 The expression $z = r(\cos\varphi + i\sin\varphi)$ is called a *trigonometric form of a complex number***.**

The transition from the algebraic to the trigonometric form is determined by the relations:

$$r = \sqrt{x^2 + y^2}\,;\quad \cos\varphi = \frac{x}{\sqrt{x^2 + y^2}}\,;\quad \sin\varphi = \frac{y}{\sqrt{x^2 + y^2}}\,.$$

Definition 3.7 If we turn to *Euler's basic formula*

$$e^{i\varphi} = \cos\varphi + i\sin\varphi,$$

then it is possible to pass from the trigonometric form to the *exponential form* of a complex number $z = re^{i\varphi}$.

Example 3.2 Plot on a complex plane and present in trigonometric and exponential forms the following complex numbers given in algebraic form:

$$z_1 = -\sqrt{3} - i\,;\quad z_2 = 2 - 2i\,;\quad z_3 = 2i\,;\quad z_4 = -2\,;\quad z_5 = -2 + i\,.$$

Solution. Draw the given numbers on the complex plane (fig. 3.3). Find the modulus and principal value of the argument of each of these numbers and write them in trigonometric and exponential forms:

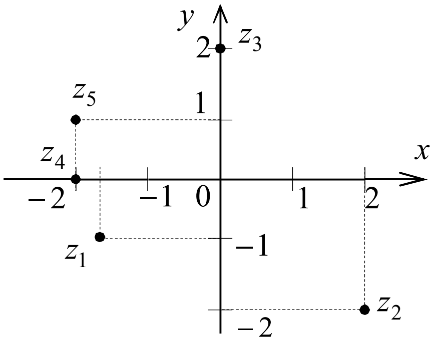


Figure 3.3

$\underline{z_1 = -\sqrt{3} - i:}$

$$x_1 = -\sqrt{3}\,;\quad y_1 = -1\,;$$

$$|z_1| = \sqrt{x_1^2 + y_1^2} = 2\,;$$

$$\arg z_1 = \operatorname{arctg}\left(y_1/x_1\right) - \pi,$$

$$x_1 < 0, y_1 < 0\,;$$

$$\arg z_1 = \operatorname{arctg}\left(1/\sqrt{3}\right) - \pi$$

$$\arg z_1 = \pi/6 - \pi = -5\pi/6;$$

$$z_1 = 2\left(\cos(-5\pi/6) + i\sin(-5\pi/6)\right); \quad z_1 = 2e^{i(-5\pi/6)}; \underline{z_2 = 2 - 2i}:$$

$$x_2 = 2; \quad y_2 = -2; \quad |z_2| = \sqrt{x_2^2 + y_2^2} = 2\sqrt{2};$$

$$\arg z_2 = \operatorname{arctg}(y_2/x_2), \; x_2 > 0; \; \arg z_2 = \operatorname{arctg}(-1) = -\pi/4;$$

$$z_2 = 2\sqrt{2}\left(\cos(-\pi/4) + i\sin(-\pi/4)\right); \quad z_2 = 2\sqrt{2}\,e^{-i(\pi/4)}.$$

$$\underline{z_3 = 2i}: \quad x_3 = 0; \quad y_3 = 2; \quad |z_3| = \sqrt{x_3^2 + y_3^2} = 2;$$

$$\arg z_3 = \pi/2, \; x = 0; \, y > 0;$$

$$z_3 = 2\left(\cos(\pi/2) + i\sin(\pi/2)\right); \quad z_3 = 2e^{i(\pi/2)}.$$

$$\underline{z_4 = -2}: \quad x_4 = -2; \quad y_4 = 0; \quad |z_4| = \sqrt{x_4^2 + y_4^2} = 2;$$

$$\arg z_4 = \operatorname{arctg}(y_4/x_4) + \pi, \; x_4 < 0, y_4 \ge 0;$$

$$\arg z_4 = \operatorname{arctg} 0 + \pi = \pi; \quad z_4 = 2(\cos\pi + i\sin\pi); \quad z_4 = 2e^{i\pi}.$$

$$\underline{z_5 = -2 + i}: \quad x_5 = -2; \quad y_5 = 1; \quad |z_5| = \sqrt{x_5^2 + y_5^2} = \sqrt{5};$$

$$\arg z_5 = \operatorname{arctg}(y_5/x_5) + \pi, \; x_5 < 0, y_5 \ge 0;$$

$$\arg z_5 = \operatorname{arctg}(-1/2) +$$

$$+ \pi = \pi - \operatorname{arctg}(1/2); \quad z_5 = \sqrt{5}\,e^{i(\pi - \operatorname{arctg}(1/2))};$$

$$z_5 = \sqrt{5}\left(\cos(\pi - \operatorname{arctg}(1/2)) + i\sin(\pi - \operatorname{arctg}(1/2))\right).$$

Definition 3.8 If $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$ are two complex number in a trigonometric form, then their product is:

$$z_1 z_2 = r_1(\cos\varphi_1 + i\sin\varphi_1) r_2(\cos\varphi_2 + i\sin\varphi_2) =$$

$$= r_1 r_2\left(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\right) =$$

$$= r_1 r_2\left(\cos\varphi_1 \cos\varphi_2 + i\cos\varphi_1 \sin\varphi_2 + i\sin\varphi_1 \cos\varphi_2 +\right.$$

$$\left.+ i^2 \sin\varphi_1 \sin\varphi_2\right) = r_1 r_2\left(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\right).$$

Definition 3.9 The product of two complex number z_1 and z_2, is a complex number which module equals product of modules, and its argument is a sum of the multiplier's arguments. So,

$$z_1 z_2 = r_1 r_2\left(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\right); \quad z_1 z_2 = r_1 r_2\, e^{(\varphi_1 + \varphi_2)i};$$

$$|z_1 z_2| = |z_1| \cdot |z_2|; \quad Arg(z_1 z_2) = Arg\, z_1 + Arg\, z_2 \, .$$

Definition 3.10 If $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$ are two complex numbers in a trigonometric form, at the same time z_2 doesn't equal zero $z_2 \neq 0$, the their division is:

$$\frac{z_1}{z_2} = \frac{r_1(\cos\varphi_1 + i\sin\varphi_1)}{r_2(\cos\varphi_2 + i\sin\varphi_2)} =$$

$$= \frac{r_1}{r_2} \cdot \frac{(\cos\varphi_1 + i\sin\varphi_1)(\cos\varphi_2 - i\sin\varphi_2)}{(\cos\varphi_2 + i\sin\varphi_2)(\cos\varphi_2 - i\sin\varphi_2)} =$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot \left(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\right).$$

Definition 3.11 The division z_1/z_2 of two complex numbers z_1 i z_2, when $z_2 \neq 0$, is a complex number, module of it is a division of the given complex numbers z_1 and z_2, and. the argument is a subtraction of z_1 and z_2. So,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\right); \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{(\varphi_1 - \varphi_2)i};$$

$$|z_1/z_2| = |z_1|/|z_2|; \qquad Arg\,(z_1/z_2) = Arg\,z_1 - Arg\,z_2.$$

Definition 3.12 *The natural power* z^n of a complex number z is a complex number obtained by multiplying the number by itself n times, where n is a natural number.

The first formula of Muavra follows from the rule of multiplication of complex numbers in trigonometric form:

$$z^n = \left(r \cdot (\cos\varphi + i\sin\varphi)\right)^n = r^n(\cos n\varphi + i\sin n\varphi).$$

The root of the n-th power $\sqrt[n]{z}$ of a complex number z is a complex number whose n-th power is equal to n-th power of z:

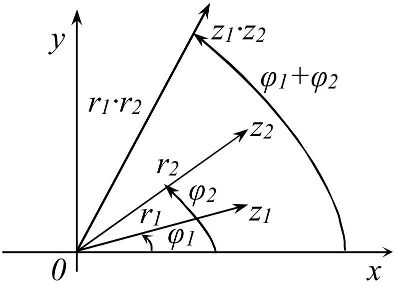


Figure 3.4

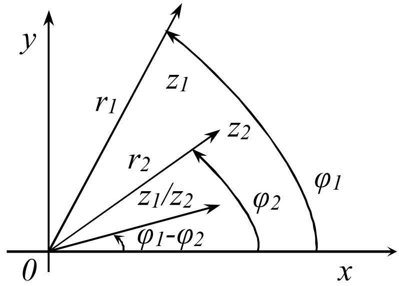


Figure 3.5

Remark 3.10 Obviously, the root of the n-th power from zero is equal to zero.

If a complex number z doesn't equal zero $z \neq 0$, then the root

n-th power $\sqrt[n]{z}$ has n different values, it can be determined by ***the second formula of Muavra***:

$$\sqrt[n]{z} = \sqrt[n]{r \cdot (\cos \varphi + i \sin \varphi)} =$$

$$= \sqrt[n]{r}\left(\cos\frac{\varphi + 2\pi k}{n} + i \sin\frac{\varphi + 2\pi k}{n}\right),$$

where $k = 0,1,2,\ldots,n-1$; $\sqrt[n]{r}$ is the arithmetic value of the root of a positive number.

Remark 3.11 All roots of the n-th power $\sqrt[n]{z}$ of a complex number $z \neq 0$ on the complex plane are represented by the vertices of a regular n-angle, inscribed in a circle with center at the origin and radius $\sqrt[n]{r}$.

Remark 3.12 At least one root of the n-th power of a positive real number will be real number.

Example 3.3 Calculate: $\left(\sqrt{3} + i\right)^{10}$.

Solution. Write down the number $\sqrt{3} + i$ in a trigonometric form

$$\sqrt{3} + i = 2\left(\cos(\pi/6) + i \sin(\pi/6)\right).$$

Accordingly to the first Muavra formula

$$\left(\sqrt{3}+i\right)^{10}=\left(2\left(\cos(\pi/6)+i\sin(\pi/6)\right)\right)^{10}=$$

$$=2^{10}\left(\cos(5\pi/3)+i\sin(5\pi/3)\right)=2^{10}\left(\cos(\pi+2\pi/3)+\right.$$

$$\left.+i\sin(\pi+2\pi/3)\right)=2^{10}\left(-\cos(2\pi/3)-i\sin(2\pi/3)\right)=$$

$$=2^{10}\left(1/2-i\cdot\sqrt{3}/2\right)=2^{9}-i\cdot 2^{9}\sqrt{3}\,.$$

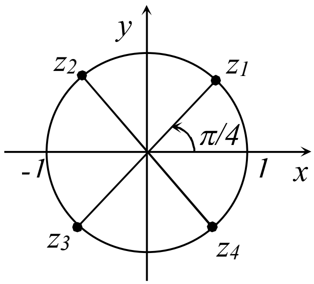


Figure 3.6

Example 3.4 Find all values of the root of the fourth degree $\sqrt[4]{-1}$.

Solution. Write down the number -1 in a trigonometric form

$$-1=1(\cos\pi+i\sin\pi)$$

(look at the fig. 3.6).

Accordingly to the second Muavra formula

$$\sqrt[4]{-1}=\sqrt[4]{1}(\cos(\pi+2\pi k)/4+$$

$$+i\sin(\pi+2\pi k)/4)\text{, where } k=0,1,2,3\,.$$

That is, the roots are complex numbers:

$$z_1=\cos(\pi/4)+i\sin(\pi/4)=\left(\sqrt{2}/2\right)(1+i)\,;$$

$$z_2=\cos(3\pi/4)+i\sin(3\pi/4)=\left(\sqrt{2}/2\right)(-1+i)\,;$$

$$z_3=\cos(5\pi/4)+i\sin(5\pi/4)=\left(\sqrt{2}/2\right)(-1-i)\,;$$

$$z_4=\cos(7\pi/4)+i\sin(7\pi/4)=\left(\sqrt{2}/2\right)(1-i)\,,$$

which are shown at the figure 3.6.

The function of a complex variable

Definition 3.13 It

$$P_n(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$$

is called *a polynomial of n-th power* of a standard form.

There z is a complex argument: n-th power polynomial; a_0, a_1, ..., a_n are constant *complex coefficients*; a_0 is called *the highest coefficient*, so $a_0 \neq 0$; a_n is called *the free item*.

Theorem 3.1 (Bezu's theorem). *The remainder of the division of polynomial $P_n(z)$ by the subtraction $z - a$ is equal to $P_n(a)$.*

Proof. $P_n(z) = Q_{n-1}(z) \cdot (z - a) + R$. Let $z \to a$, then $P_n(a) = R$. That we need to get.

Corollary 3. 1 If a is a root of a polynomial $P_n(z)$, then the polynomial $P_n(z)$ is divided without remainder by the subtraction $z - a$, that is, it decomposes into factors

$$P_n(z) = Q_{n-1}(z) \cdot (z - a),$$

where a quotient $Q_{n-1}(z)$ is a polynomial of power one less power.

Theorem 3.2 (the base algebra theorem). *Any polynomial $P_n(z)$ of nonzero power $n \geq 1$ has at least one root (real or complex).*

***Corollary* 3.2** Any polynomial $P_n(z)$ of nonzero power $n \geq 1$ has n roots, among them may be the equal roots.

***Corollary* 3.3** Any polynomial $P_n(z)$ of nonzero power $n \geq 1$ decomposes into factors in the form:

$$P_n(z) = a_0 (z - z_1)^{k_1} (z - z_2)^{k_2} \ldots (z - z_m)^{k_m},$$

where a_0 is the highest coefficient; $z_1, z_2, \ldots, z_m$ are different roots (*real or complex*); $k_1, k_2, \ldots, k_m$ are corresponding multiplicities of these roots, and $k_1 + k_2 + \ldots + k_m = n$.

The roots of the quadratic equation $az^2+bz+c=0 \quad (a \neq 0)$ with complex coefficients a, b, c can be found by:

$$z_{1,2}=\frac{-b\pm\sqrt{D}}{2a}; \quad D=b^2-4ac,$$

where $\sqrt{D}$ – one of the values of the square root of the discriminant D.

Vieta's theorem remains valid on the set of complex numbers for the roots of a quadratic equation:

$$z_1+z_2=-b/a, \quad z_1 z_2=c/a.$$

Example 3.5 Solve the quadratic equation $4z^2-8z+5=0$.

Solution. $D=8^2-4\cdot 4\cdot 5=-16$; $\sqrt{D}=\sqrt{-16}=4i$;

$$z_{1,2}=\frac{8\pm 4i}{2\cdot 4}=1\pm\frac{1}{2}i.$$

The complex function z of a real variable t of a real variable t з from some nonempty set D of real numbers according to a certain law corresponds to a single value of a complex variable z from some area E of the complex plane. The complex function $z=z(t)$ of a real variable t is determined by the equality $z=x(t)+i\,y(t)$, $t\in D$, where $x(t)$ and $y(t)$ are the given real functions (respectively real and imaginary parts of the variable $z=z(t)$).

The function $z=z(t)$, $t\in[\alpha;\beta]$ in a complex-parametric form sets some flat line L. Parametric equations of this line are: $x=x(t)$, $y=y(t)$, $t\in[\alpha;\beta]$.

The complex variable $z=z(t)$ corresponds to a vector function.

To find the $z'=z'(t)$ of a complex function $z=x(t)+i\,y(t)$ of a real variable, it is necessary to differentiate separately the real $x(t)$ and imaginary $y(t)$ parts: $z'=x'(t)+i\,y'(t)$.

Example 3.6 Determine the form and draw on the complex plane the line given by the equation $z = 3\cos t + 3i\sin t$.

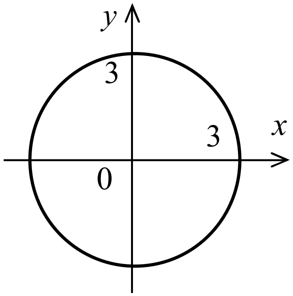


Figure 3.7

Solution. To determine the type of line, substitute in its equation $z = x + iy$ and reduce it to the corresponding standard form. Then let's draw this line.

$$x + iy = 3\cos t + 3i\sin t;$$

$$\begin{cases} x = 3\cos t, \\ y = 3\sin t. \end{cases}$$

is a circle with a radius $r = 3$ centered at origin (look at the appendix A), given in parametric form (figure 3.7). It can be implicitly given by equations $|z| = 3$. Canonical equation of a circle is

$$x^2 + y^2 = 9.$$

Example 3.7 Calculate the value of the function $\cos(3 + 4i)$.

Solution.

$$\cos(3+4i) = \frac{e^{i(3+4i)} + e^{-i(3+4i)}}{2} = \frac{e^{-4}e^{3i} + e^{4}e^{-3i}}{2} =$$

$$= \left(e^{-4}(\cos 3 + i\sin 3) + e^{4}(\cos 3 - i\sin 3)\right)/2 =$$

$$= \left(\cos 3\cdot(e^{-4} + e^{4}) - i\sin 3\cdot(e^{4} - e^{-4})\right)/2 =$$

$$= \frac{e^{4} + e^{-4}}{2}\cos 3 - i\frac{e^{4} - e^{-4}}{2}\sin 3 = ch\,4\cos 3 - i\,sh\,4\sin 3.$$

Example 3.8 Determine the form and plot the lines given by the equations on the complex plane:

a) $|z - 2 + i| = 1$; b) $z = (1 + 2i)\cdot t$.

Solution. To determine the type of line, substitute in its equation and reduce it to the appropriate standard form. Then let's draw these lines.

a) $|x+iy-2+i|=1$; $|(x+2)+i(y+1)|=1$;

$\sqrt{(x+2)^2+(y+1)^2}=1$; $(x+2)^2+(y+1)^2=1$ is a circle with radius $R=1$ and it`s centered at a point $O(2,-1)$;

b) $x + iy = (1 + 2i)\, t \Rightarrow x + iy = t + 2it \Rightarrow$;

$\Rightarrow \begin{cases} x = t \\ y = 2t \end{cases}$ is a straight line given in parametric form, its explicit equation is $y=2x$.

Draw the lines graphs by yourself.

Lecture 4 THEORY OF LIMITS

Definition 4.1 The number A is called *the limit of the function* $y = f(x)$ when $x \to a$, if for all values of x that differ little enough from the number a, the corresponding values of the function $y = f(x)$ differ little enough from the number A:

$$\lim_{x \to a} f(x) = A.$$

If $x < a$ and $x \to a$, then we write conventionally $x \to a - 0$; similarly, if $x > a$ and $x \to a$, then we write $x \to a + 0$. The numbers

$$f(a-0) = \lim_{x \to a-0} f(x) \text{ and } f(a+0) = \lim_{x \to a+0} f(x)$$

are called, respectively, *the limit of the function* $f(x)$ *from the left* and *the limit of the function* $f(x)$ *from the right* at the point a (if these numbers exist).

For the existence of the limit of a function $f(x)$ as $x \to a$, it is necessary and sufficient to have the following equality:

$$f(a-0) = f(a+0).$$

Example 4.1 Compute the limits on the right and left of the function

$$f(x) = \arctan\frac{1}{x}$$

when $x \to 0$.

Solution:

$$f(+0) = \lim_{x \to +0}\left(\arctan\frac{1}{x}\right) = \arctan\frac{1}{+0} = \arctan(+\infty) = \frac{\pi}{2},$$

$$f(-0) = \lim_{x \to -0}\left(\arctan\frac{1}{x}\right) = \arctan\frac{1}{-0} = \arctan(-\infty) = -\frac{\pi}{2}.$$

Obviously, the function $f(x)$ in this case has no limit as $x \to 0$.

If the limits $\lim\limits_{x\to a} f_1(x)$ and $\lim\limits_{x\to a} f_2(x)$ exist, then the following theorems hold:

1) $\lim\limits_{x\to a}\left[f_1(x)\pm f_2(x)\right]=\lim\limits_{x\to a} f_1(x)\pm\lim\limits_{x\to a} f_2(x)$,

2) $\lim\limits_{x\to a}\left[f_1(x)\cdot f_2(x)\right]=\lim\limits_{x\to a} f_1(x)\cdot\lim\limits_{x\to a} f_2(x)$,

3) $\lim\limits_{x\to a}\left[C\cdot f_1(x)\right]=C\cdot\lim\limits_{x\to a} f_1(x)$,

4) $\lim\limits_{x\to a}\dfrac{f_1(x)}{f_2(x)}=\dfrac{\lim\limits_{x\to a} f_1(x)}{\lim\limits_{x\to a} f_2(x)}$ ($\lim\limits_{x\to a} f_2(x)\neq 0$).

Example 4.2 Compute $\lim\limits_{x\to -2}\dfrac{x^2-x+4}{x^2+1}$.*Solution:*

$$\lim_{x\to -2}\frac{x^2-x+4}{x^2+1}=\frac{\lim\limits_{x\to -2}\left(x^2-x+4\right)}{\lim\limits_{x\to -2}\left(x^2+1\right)}=$$

$$\frac{\lim\limits_{x\to -2} x^2+\lim\limits_{x\to -2} x+\lim\limits_{x\to -2} 4}{\lim\limits_{x\to -2} x^2+\lim\limits_{x\to -2} 1}=\frac{\left(\lim\limits_{x\to -2} x\right)^2+\lim\limits_{x\to -2} x+\lim\limits_{x\to -2} 4}{\left(\lim\limits_{x\to -2} x\right)^2+\lim\limits_{x\to -2} 1}=$$

$$=\frac{\left(\lim\limits_{x\to -2} x\right)^2+\lim\limits_{x\to -2} x+\lim\limits_{x\to -2} 4}{\left(\lim\limits_{x\to -2} x\right)^2+\lim\limits_{x\to -2} 1}=\frac{\left(\lim\limits_{x\to -2}(-2)\right)^2+\lim\limits_{x\to -2}(-2)+\lim\limits_{x\to -2} 4}{\left(\lim\limits_{x\to -2}(-2)\right)^2+\lim\limits_{x\to -2} 1}=$$

$$=\frac{(-2)^2+(-2)+4}{(-2)^2+1}=\frac{6}{5}.$$

Definition 4.2 The function $f(x)$ is called *infinitesimal* as $x \to a$ if

$$\lim_{x \to a} f(x) = 0.$$

Definition 4.3 The function $f(x)$ is called *infinitude* as $x \to a$ if

$$\lim_{x \to a} f(x) = \infty.$$

Properties of infinitesimal and infinitude functions:

1) if $f(x)$ is infinitesimal function as $x \to a$, then $-f(x)$ is also infinitesimal one;

2) if $f_1(x)$ and $f_2(x)$ are infinitesimal functions as $x \to a$, then $f_1(x) \pm f_2(x)$ is also infinitesimal one;

3) if $f_1(x)$ and $f_2(x)$ are infinitude functions as $x \to a$, then $f_1(x) + f_2(x)$ and $f_1(x) \cdot f_2(x)$ are also infinitude ones;

4) if $\lim\limits_{x \to a} f_1(x) = b = \text{const}$, $\lim\limits_{x \to a} f_2(x) = \infty$, then

$$\lim_{x \to a} \left[f_1(x) + f_2(x) \right] = b + \infty = \infty, \quad \lim_{x \to a} \left[f_1(x) \cdot f_2(x) \right] = b \cdot \infty = \infty,$$

$$\lim_{x \to a} \left(f_2(x) \right)^{f_1(x)} = \infty^b = \infty, \quad \lim_{x \to a} \sqrt[f_1(x)]{f_2(x)} = \sqrt[b]{\infty} = \infty,$$

$$\lim_{x \to a} \frac{f_1(x)}{f_2(x)} = \frac{b}{\infty} = 0;$$

5) if $\lim\limits_{x \to a} f_1(x) = b = \text{const}$, $\lim\limits_{x \to a} f_2(x) = 0$, then

$$\lim_{x \to a} \frac{f_1(x)}{f_2(x)} = \frac{b}{0} = \infty.$$

We will now consider the cases where, for some assigned value of x, the numerator and denominator are both zero or both infinities. The fraction is then said to be *indeterminate.*

During the computing of the limit of two integral polynomials ratio as $x \to \infty$ and getting the indeterminate form $\left[\dfrac{\infty}{\infty}\right]$, it is necessary, firstly, to divide both terms of the ratio by x^n, where n is the highest power of these polynomials. A similar procedure is also possible in many cases for fractions containing irrational terms.

Example 4.3 Compute $\lim\limits_{x\to\infty}\dfrac{x^2-3x+5}{2x^2-3}$.

Solution:

$$\lim_{x\to\infty}\frac{x^2-3x+5}{2x^2-3}=\frac{\infty^2-3\cdot\infty+5}{2\cdot\infty^2-3}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\frac{x^2}{x^2}-\frac{3x}{x^2}+\frac{5}{x^2}}{\frac{2x^2}{x^2}-\frac{3}{x^2}}=\lim_{x\to\infty}\frac{1-\frac{3}{x}+\frac{5}{x^2}}{2-\frac{3}{x^2}}=$$

$$=\frac{1-\frac{3}{\infty}+\frac{5}{\infty^2}}{2-\frac{3}{\infty^2}}=\frac{1-0+0}{2-0}=\frac{1}{2}.$$

Example 4.4 Compute $\lim\limits_{x\to\infty}\dfrac{x^2-3x+5}{x-3}$.

Solution:

$$\lim_{x\to\infty}\frac{x^2-3x+5}{x-3}=\frac{\infty^2-3\cdot\infty+5}{\infty-3}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\frac{x^2}{x^2}-\frac{3x}{x^2}+\frac{5}{x^2}}{\frac{x}{x^2}-\frac{3}{x^2}}=\lim_{x\to\infty}\frac{1-\frac{3}{x}+\frac{5}{x^2}}{\frac{1}{x}-\frac{3}{x^2}}=$$

$$=\frac{1-\frac{3}{\infty}+\frac{5}{\infty^2}}{\frac{1}{\infty}-\frac{3}{\infty^2}}=\frac{1-0+0}{0-0}=\frac{1}{0}=\infty.$$

If $P(x)$ and $Q(x)$ are integral polynomials and $P(a) \neq 0$ or $Q(a) \neq 0$, then the limit of the rational fraction

$$\lim_{x \to a} \frac{P(x)}{Q(x)}$$

is obtained directly.

But if $P(a) = Q(a) = 0$, then it is advisable to cancel the binomial $x = -a$ out of the fraction $\frac{P(x)}{Q(x)}$ once or several times. To do it, we can use the formulas of abridged multiplication:

1) $a^2 - b^2 = (a-b)(a+b)$,

2) $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$,

3) $ax^2 + bx + c = a(x - x_1)(x - x_2)$, where x_1, x_2 are roots of the equation $ax^2 + bx + c = 0$ which can be found by using the discriminant:

$$D = b^2 - 4ac, \; x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}.$$

Example 4.5 Evaluate the following limits

$$\text{a) } \lim_{x \to -1} \frac{2x^2 + x - 1}{x^2 - 1}; \text{ b) } \lim_{x \to 3} \frac{2x^2 - 5x - 3}{x^3 - 27}.$$

Solution:

a) $\lim_{x \to -1} \frac{2x^2 + x - 1}{x^2 - 1} = \frac{2 \cdot (-1)^2 - 1 - 1}{(-1)^2 - 1} = \frac{2 - 1 - 1}{1 - 1} = \left[\frac{0}{0}\right] =$

factorize the numerator and denominator and cancel:

$$2x^2 + x - 1, \quad x^2 - 1 = (x-1)(x+1),$$

$$D = 1^2 - 4 \cdot 2 \cdot (-1) = 1 + 8 = 9,$$

$$x_1 = \frac{-1+\sqrt{9}}{2\cdot 2} = \frac{-1+3}{4} = \frac{2}{4} = \frac{1}{2},$$

$$x_2 = \frac{-1-\sqrt{9}}{2\cdot 2} = \frac{-1-3}{4} = \frac{-4}{4} = -1,$$

$$2x^2 + x - 1 = 2\left(x - \frac{1}{2}\right)(x+1).$$

$$= \lim_{x\to -1} \frac{2\left(x-\frac{1}{2}\right)(x+1)}{(x-1)(x+1)} = \lim_{x\to -1} \frac{2\left(x-\frac{1}{2}\right)}{x-1} = \lim_{x\to -1} \frac{2x-1}{x-1} = \frac{2\cdot(-1)-1}{-1-1} = \frac{-2-1}{-1-1} = \frac{-3}{-2} = \frac{3}{2};$$

b) $\lim\limits_{x\to 3} \dfrac{2x^2-5x-3}{x^3-27} = \left|\dfrac{0}{0}\right| =$

$$= \left| \begin{array}{c} 2x^2 - 5x - 3 = 2(x-3)(x+1/2) \\ D = 1; x_1 = 3; x_2 = -\dfrac{1}{2} \\ x^3 - 27 = (x-3)(x^2+3x+9) \end{array} \right| = \lim_{x\to 3} \frac{2(x-3)(x+1/2)}{(x-3)(x^2+3x+9)} =$$

$$= \lim_{x\to 3} \frac{2(x+1/2)}{x^2+3x+9} = \frac{2\cdot 3+1}{9+9+9} = \frac{7}{27}.$$

To find the limit of an irrational expression, when one gets the indeterminate value $\left[\dfrac{0}{0}\right]$ or $[\infty-\infty]$, it is necessary to transfer the irrational term from the numerator to the denominator, or vice versa, from the denominator to the numerator.

Example 4.6 Compute $\lim\limits_{x\to 0} \dfrac{x}{\sqrt{1+3x}-1}$.

Solution.

$$\lim_{x\to 0} \frac{x}{\sqrt{1+3x}-1} = \frac{0}{\sqrt{1+3\cdot 0}-1} = \left[\frac{0}{0}\right] =$$

Multiply the numerator and denominator of the fraction under the limit sign by the expression conjugate of the denominator, i.e. by $\sqrt{1+3x}+1$:

$$\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{\left(\sqrt{1+3x}-1\right)\left(\sqrt{1+3x}+1\right)}=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{\left(\sqrt{1+3x}\right)^2-1^2}=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{(1+3x-1)}=$$

$$=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{3x}=\lim_{x\to 0}\frac{\sqrt{1+3x}+1}{3}=\frac{\sqrt{1+0}+1}{3}=\frac{2}{3}.$$

We have two fundamental limits that help us to simplify the limits calculation and they are frequently used.

Definition 4.4 *The first fundamental limit is*:

$$\boxed{\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha}=1},$$

and some useful consequences from it:

$$\lim_{\alpha\to 0}\frac{\tan\alpha}{\alpha}=1,\quad \lim_{\alpha\to 0}\frac{\arcsin\alpha}{\alpha}=1,\quad \lim_{\alpha\to 0}\frac{\arctan\alpha}{\alpha}=1,$$

$$\lim_{\alpha\to 0}\frac{\alpha}{\sin\alpha}=1,\quad \lim_{\alpha\to 0}\frac{\alpha}{\tan\alpha}=1,\quad \lim_{\alpha\to 0}\frac{\alpha}{\arcsin\alpha}=1,$$

$$\lim_{\alpha\to 0}\frac{\alpha}{\arctan\alpha}=1.$$

Definition 4.5 *The second fundamental limit is*:

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e\approx 2,72}.$$

If you get the indeterminate value $\left[\frac{0}{0}\right]$ of the limit with trigonometric expressions, it is necessary to factorize the numerator and denominator by using trigonometric formulas and cancel or apply the frequently used limits for trigonometric functions.

Remark 4.1 Sometimes we need to use your school knowledge about the trigonometric functions which we can find at Appendices C, D.

Example 4.7 Compute $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}$.

Solution:

$$\lim_{x\to 0}\frac{\sin 3x}{x}=\frac{\sin 0}{0}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\sin 3x\cdot 3}{x\cdot 3}=3\cdot\lim_{x\to 0}\frac{\sin 3x}{3x}=3\cdot 1=3.$$

Example 4.8 Compute: a) $\lim\limits_{x\to 0}\dfrac{1-\cos 4x}{1-\cos 8x}$, b) $\lim\limits_{x\to\pi/4}\dfrac{ctg(\pi/4+x)}{4x-\pi}$.

Solution:

$$\text{a) }\lim_{x\to 0}\frac{1-\cos 4x}{1-\cos 8x}=\frac{1-\cos 0}{1-\cos 0}=\left[\frac{0}{0}\right]=,$$

firstly, we should use a trigonometric formula $\sin^2\alpha=\dfrac{1-\cos 2\alpha}{2}$ (look at Appendices B, D), then frequently used limits for trigonometric functions:

$$=\left|1-\cos 2\alpha=2\sin^2\alpha\right|=\lim_{x\to 0}\frac{2\sin^2\frac{4x}{2}}{2\sin^2\frac{8x}{2}}=\lim_{x\to 0}\frac{\sin^2 2x}{\sin^2 4x}=\lim_{x\to 0}\frac{\sin 2x\cdot\sin 2x\cdot 2x\cdot 2x}{\sin^2 4x\cdot 2x\cdot 2x}=$$

$$=\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\lim_{x\to 0}\frac{4x^2}{\sin^2 4x}=1\cdot 1\cdot\lim_{x\to 0}\frac{4x}{\sin 4x}\cdot\lim_{x\to 0}\frac{x}{\sin 4x}=$$

$$=1\cdot 1\cdot 1\cdot\lim_{x\to 0}\frac{x\cdot 4}{\sin 4x\cdot 4}=\lim_{x\to 0}\frac{x\cdot 4}{\sin 4x}\cdot\frac{1}{4}=1\cdot\frac{1}{4}=\frac{1}{4};$$

$$\text{b) }\lim_{x\to\pi/4}\frac{ctg(\pi/4+x)}{4x-\pi}=\left|\frac{0}{0}\right|=\left|\begin{matrix}u=x-\pi/4;\ x=\pi/4+u;\\ x\to\pi/4\ \Rightarrow\ u\to 0\end{matrix}\right|=$$

$$\lim_{u\to 0}\frac{ctg(\pi/4+u+\pi/4)}{4(\pi/4+u)-\pi}=\lim_{u\to 0}\frac{ctg(\pi/2+u)}{4u}=$$

$$=-\frac{1}{4}\cdot\lim_{u\to 0}\frac{tg\,u}{u}=-\frac{1}{4}\cdot 1=-\frac{1}{4}.$$

When taking limits of the form

$$\lim_{x\to a}\left[f(x)\right]^{g(x)}=A,$$

one should bear in mind that:

1) if there are final limits

$$\lim_{x\to a}f(x)=B \text{ and } \lim_{x\to a}g(x)=C,$$

then $A=B^C$;

2) if $\lim_{x\to a}f(x)=B\neq 1$ and $\lim_{x\to a}g(x)=+\infty$,

then

$$A=\begin{cases}0, & B<1;\\ \infty, & B>1;\end{cases}$$

3) if $\lim_{x\to a}f(x)=B=1$ and $\lim_{x\to a}g(x)=\infty$, then we get the indefinite value $\left[1^{\infty}\right]$ and should use frequently used limits for exponential functions.

Example 4.9 Compute $\lim_{x\to\infty}\left(\frac{2x+3}{x-1}\right)^{4x}$.

Solution:

$$f(x)=\frac{2x+3}{x-1},\quad g(x)=4x,$$

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{2x+3}{x-1}=\frac{2\cdot\infty+3}{\infty-1}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\frac{2x}{x}+\frac{3}{x}}{\frac{x}{x}-\frac{1}{x}}=\lim_{x\to\infty}\frac{2+\frac{3}{x}}{1-\frac{1}{x}}=\frac{2+\frac{3}{\infty}}{1-\frac{1}{\infty}}=\frac{2+0}{1-0}=2,$$

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty}(4x) = 4\cdot\infty = \infty .$$

Thus, we have the second case and $\lim_{x\to\infty}\left(\frac{2x+3}{x-1}\right)^{4x} = 2^{\infty} = \infty$, because $2 > 1$.

Example 4.10 Compute $\lim_{x\to\infty}\left(\frac{x+3}{x-1}\right)^{4x}$.

Solution:

$$f(x) = \frac{x+3}{x-1}, \quad g(x) = 4x,$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{x+3}{x-1} = \frac{\infty+3}{\infty-1} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty}\frac{\frac{x}{x}+\frac{3}{x}}{\frac{x}{x}-\frac{1}{x}} = \lim_{x\to\infty}\frac{1+\frac{3}{x}}{1-\frac{1}{x}} = \frac{1+\frac{3}{\infty}}{1-\frac{1}{\infty}} = \frac{1+0}{1-0} = 1,$$

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty}(4x) = 4\cdot\infty = \infty .$$

Thus, we have the third case: $\lim_{x\to\infty}\left(\frac{x+3}{x-1}\right)^{4x} = \left[1^{\infty}\right] =$

and to find the limit we should use frequently used limits for exponential functions:

$$= \lim_{x\to\infty}\left(1+\left[\frac{x+3}{x-1}-1\right]\right)^{4x} = \lim_{x\to\infty}\left(1+\frac{x+3-(x-1)}{x-1}\right)^{4x} = \lim_{x\to\infty}\left(1+\frac{x+3-x+1}{x-1}\right)^{4x} = \lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{4x} =$$

$$= \lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{4x\cdot\frac{4}{x-1}\cdot\frac{x-1}{4}} = \lim_{x\to\infty}\left(\left(1+\frac{4}{x-1}\right)^{\frac{x-1}{4}}\right)^{4x\cdot\frac{4}{x-1}} = e^{\lim\limits_{x\to\infty}\frac{16x}{x-1}} = e^{\frac{16\cdot\infty}{\infty-1}} = e^{\left[\frac{\infty}{\infty}\right]} =$$

$$= e^{\lim\limits_{x\to\infty}\frac{\frac{16x}{x}}{\frac{x}{x}-\frac{1}{x}}} = e^{\lim\limits_{x\to\infty}\frac{16}{1-\frac{1}{x}}} = e^{\frac{16}{1-\frac{1}{\infty}}} = e^{\frac{16}{1-0}} = e^{16} .$$

Table 4.1 – Equivalent infinitely small values

$\sin x \underset{x\to 0}{\sim} x$	$arctg\, x \underset{x\to 0}{\sim} x$	$a^x - 1 \underset{x\to 0}{\sim} x \ln a$
$tg\, x \underset{x\to 0}{\sim} x$	$1-\cos x \underset{x\to 0}{\sim} x^2/2$	$\ln(1+x) \underset{x\to 0}{\sim} x$
$\arcsin x \underset{x\to 0}{\sim} x$	$e^x - 1 \underset{x\to 0}{\sim} x$	$(1+x)^\alpha - 1 \underset{x\to 0}{\sim} \alpha x$

Example 4.11 Evaluate the following limits:

$$1)\ \lim_{x\to 0}\frac{\ln\left(1-6x^2\right)}{arctg\, x}; \qquad 2)\ \lim_{x\to 0}\frac{\arcsin 3x - x^2}{1-\sqrt[7]{1+x}}.$$

Solution: 1) $\lim_{x\to 0}\frac{\ln\left(1-6x^2\right)}{arctg\, x} = \left|\frac{0}{0}\right| = \left|\ln\left(1-6x^2\right) \sim -6x^2;\right.$

$$\left. arctg\, x \sim x \right| = \lim_{x\to 0}\frac{-6x^2}{x} = 0;$$

$$2)\ \lim_{x\to 0}\frac{\arcsin 3x - x^2}{1-\sqrt[7]{1+x}} = \left|\frac{0}{0}\right| = \left|\arcsin 3x \sim 3x;\right.$$

$$\left. 1-\sqrt[7]{1+x} \sim -\frac{x}{7}\right| = \lim_{x\to 0}\frac{3x-x^2}{-x/7} = \lim_{x\to 0}\frac{x(3-x)}{-x/7} = \lim_{x\to 0}\frac{(3-x)}{-1/7} = -21.$$

Lecture 5 **The DERIVATIVE OF FUNCTION. TECHNIQUES of DIFFERENTIATION**

Let consider function $f(x)$ identified on the interval $(a;b)$ and $x_0 \in (a;b)$. We choose an arbitrary point x belonging to the graph of the function $f(x)$, then the increment of the argument will be called expression as $\Delta x = x - x_0$. Since the point x_0 is fixed, then the increment of the function will have the form $\Delta y = f(x_0 + \Delta x) - f(x_0)$ and depend on Δx. We will compose the ratio the increment Δy of the function to the increment of the argument Δx, when the Δx approaches to zero and we also will find the limit of its ratio as $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. This is such an important limit, and it arises in so many places that we give it a name. We call it a derivative. Here is the official definition of the derivative. On the other words, the derivative is a slope of a curve at point, it is formula.

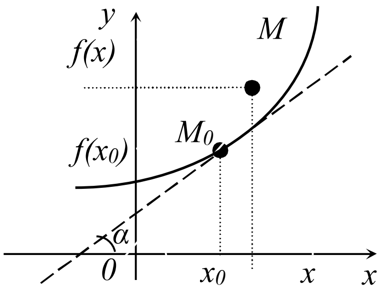


Figure 5.1

Definition 5.1 The derivative of a function $f(x)$ with respect x is the function $f'(x)$ and defined as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x)$$

Definition 5.2 The derivative of a function $f(x)$ is denoted $f'(x)$

and we often read as "f prime of x".

The geometric meaning of the derivative consists of the fact that the derived function for each value is equal to the angular coefficient of the tangent line (Figure 5.1) to the graph of this function at the corresponding point M_0, as $f'(x) = tg\alpha$; where α – is the angle that forms the tangent line to the graph with the positive direction of the abscissa; this angle is an argument of function is x. The equation of the tangent line to the graph of the function $y = f(x)$ has the following form:

$$y - y_0 = y'(x_0)(x - x_0),$$

where x_0 is abscissa of a point of tangency M_0, y_0 the corresponding ordinate of a point M_0, $y'(x_0)$ – is derivative of the function $y = f(x)$ computed at the point M_0, (and also $y'(x_0) = k$, where k – angular coefficient of tangent); y, x arbitrary variables.

For example, find the equation of tangent line to a curve of the function $y = x^2$ at the point $M(1/2; 1/4)$.

Solution: To find derivative of the function $y = x^2$ it will be: $y' = 2x$. Thus:

$tg\,\alpha = y'(x_0) = 2 \cdot (1/2) = 1$; $\alpha = arctg\,1 = 45^\circ$ – the angle of the slope of tangent line; $y - 1/4 = 1 \cdot (x - 1/2)$; $y = x - 1/4$ – the equation of tangent line.

Physical or mechanical content consists of the fact that the non-uniform motion of a material point is expressed by a function $s = f(t)$. This function changes in time t; the derivative $s'(t)$ is the rate of function's changes at a certain time t_0 (say: instantaneous velocity), that is $f'(t_0) = v(t_0)$, where $v(t_0)$ is a velocity of changes $s = f(t)$ at a certain time $t = t_0$. Thus, the velocity of occurrence of physical, chemical, and other processes is expressed by derivative.

We know that $f'(x)$ carries important information about the original function $f(x)$. In one example we saw that $f'(x)$ tells us which a steep (slope) of the graph of the function $f(x)$ is; in another we saw that $f'(x)$ tells us the velocity of an object if $f(x)$ tells us the position of the object at time x. As we said earlier, this same mathematical idea is useful whenever $f(x)$ represents some changing quantity and we want to know something about how it changes, or roughly, the "velocity" at which it changes. Most functions encountered in practice are built up from a small collection of "primitive" functions in a few simple ways, for example, by adding or multiplying functions together to get new, more complicated functions. To make good use of the information provided by $f'(x)$ we need to be able to compute it for a variety of such functions.

To recall the form of the limit, we sometimes say instead that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{dy}{dx}.$$

In other words, $\frac{dy}{dx}$ is another notation for the derivative, and it reminds us that it is related to an actual slope between two points. This notation is *called Leibniz notation*, after Gottfried Leibniz, who developed the fundamentals of calculus independently, at about the same time that Isaac Newton did. Again, since we often use f and $f(x)$ to mean the original function, we sometimes use $\frac{dy}{dx}$ and $\frac{df}{dx}$ to refer to the derivative. If the function $f(x)$ is written out in full we often write the last of these something like

$$f'(x) = \left(\sqrt{5x^2 - 4}\right)' \text{ or } \frac{d}{dx}\left[\sqrt{5x^2 - 4}\right]$$

with the function written to the side, instead of trying to fit it into the numerator.

Let's compute a couple of derivatives using the definition.

Example 5.1 Find the derivative of function $y = \sin x$ and calculate it at the point $y'\left(\frac{\pi}{3}\right)$.

Solution: $f(x) = \sin x$,

$$f(x+\Delta x) = \sin(x+\Delta x);\ \Delta y = f(x+\Delta x) - f(x) =$$

$$= \sin(x+\Delta x) - \sin x = 2\sin\frac{x+\Delta x - x}{2}\cos\frac{x+\Delta x + x}{2} =$$

$$2\sin\frac{\Delta x}{2}\cos\frac{2x+\Delta x}{2}.$$

Let us compose the ratio of the increment of the function to the increment of the argument $\frac{\Delta y}{\Delta x} = \frac{2\sin\frac{\Delta x}{2}\cos\frac{2x+\Delta x}{2}}{\Delta x}$ and calculate the limit:

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{2\sin\frac{\Delta x}{2}\cos\frac{2x+\Delta x}{2}}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}\cdot\lim_{\Delta x\to 0}\frac{\cos\left(x+\frac{\Delta x}{2}\right)}{\Delta x} = \cos x_0.$$

So, $(\sin x)' = \cos x$. Now we will calculate the value of derivative at point $x = \frac{\pi}{3}$ and obtain that it is $\cos\frac{\pi}{3} = \frac{1}{2}$.

Example 5.2 Determine $f'(0)$ for the function $f(x) = |x|$.

Solution. Since this problem is asking for the derivative at a specific point we'll go ahead and use that in our work. It will make our life easier and that's always a good thing. So, plug into the definition and simplify.

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{|0+\Delta x| - |0|}{\Delta x} = \lim_{\Delta x\to 0}\frac{|\Delta x|}{\Delta x}.$$

We saw a situation like this back when we were looking at limits at infinity. As in that section we can't just cancel the Δx's. We will have to look at the two one sided limits and recall that

$|\Delta x| = \begin{cases} -\Delta x, & if \;\; \Delta x < 0 \\ \Delta x, & if \;\; \Delta x > 0 \end{cases}$, in a right-hand limit we have $\lim\limits_{\Delta x \to +0} \frac{|\Delta x|}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$, in a left-hand limit we have $\lim\limits_{\Delta x \to -0} \frac{|\Delta x|}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{-\Delta x}{\Delta x} = -1$. The two one-sided limits are different and so $\lim\limits_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}$ doesn't exist. However, this is the limit that gives us the derivative that we're after. If the limit doesn't exist, then the derivative doesn't exist either.

Remark 5.1 In the last example we have seen a function for which the derivative doesn't exist at a point. This is a fact of life that we've got to be aware of. Derivatives will not always exist. Note as well that this doesn't say anything about whether the derivative exists anywhere else. In fact, the derivative of the absolute value function exists at every point except the one we just looked at $x = 0$.

The preceding discussion leads to the following definition.

Definition 5.3 A function $f(x)$ *is called differentiable* at $x = a$ if $f'(a)$ exists and if $f(x)$ is called differentiable on an interval if the derivative exists for each point in that interval.

The next theorem shows us a very nice relationship between functions that are continuous and those that are differentiable.

Theorem 5.1 If the function $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous at $x = a$

Note that this theorem does not work in reverse.

Consider $f(x) = |x|$ and take look at $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} |x| = 0 = f(0)$. So, $f(x) = |x|$ is continuous at $x = a$ but this function is not differentiable at $x = a$. In really, the function must be continuous. However, continuity is a necessary but no sufficient condition for differentiability.

We should note that computing most derivatives directly from the definition is a fairly complicated process filled with opportunities to make mistakes. So, we need to start mastering formulas and/or properties that will help us to take the derivative of many of the common functions and we won't need to resort to the definition of the derivative too often.

This does not mean however that it isn't important to know the definition of the derivative! It is an important definition that we should always know and keep in the back of our minds. It is just something that we're not going to be working with all that much.

There are a few ***important rules*** for computing derivatives of certain combinations of functions. *Derivatives of sums* are equal to the sum of derivatives so that

$$(u \pm v)' = u' \pm v'$$

In addition, if C is a constant,

$$(C \cdot u)' = C(u)', \; C = const$$

The *product rule* for differentiation states

$$(u \cdot v)' = u' \cdot v + v' \cdot u$$

Where u' denotes the derivative of u with respect to x. This derivative rule can be applied iteratively to yield derivative rules for products of three or more functions, for example,

$$(u \cdot v \cdot g)' = u' \cdot v \cdot g + v' \cdot u \cdot g + g' \cdot v \cdot u$$

The *quotient rule* for derivatives states that

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - v' \cdot u}{v^2}, \; v \neq 0.$$

Simple derivatives of some elementary functions will be presented at the table 5.1 below

Table 5.1 – Elementary function derivatives

	Function	Derivative
1	Constant	$C' = 0$
2	Powerful function	$\left(u^a\right)' = a u^{a-1} \cdot u'$
2a	x	$x' = 1$
2b	$\sqrt{u}$	$\left(\sqrt{u}\right)' = \frac{1}{2\sqrt{u}} \cdot u'$
2c	$\frac{1}{u}$	$\left(\frac{1}{u}\right)' = -\frac{1}{u^2} \cdot u'$
3	Indicative function	$\left(a^u\right)' = a^u \ln a \cdot u'$
3a	Exponent	$\left(e^u\right)' = e^u \cdot u'$
4	Logarithmic function	$\left(\log_a u\right)' = \frac{1}{u \ln a} \cdot u'$
4a	Natural Logarithm	$\left(\ln u\right)' = \frac{1}{u} \cdot u'$
5	Sine	$\left(\sin u\right)' = \cos u \cdot u'$
6	Cosine	$\left(\cos u\right)' = -\sin u \cdot u'$
7	Tangent	$\left(tg\ u\right)' = \frac{1}{\cos^2 u} \cdot u'$
8	Cotangent	$\left(ctg\ u\right)' = -\frac{1}{\sin^2 u} \cdot u'$

Continued Table 5.1

	Function	Derivative
9	Arcsine	$(\arcsin u)' = \frac{1}{\sqrt{1-u^2}} \cdot u'$
10	Arccosine	$(\arccos u)' = -\frac{1}{\sqrt{1-u^2}} \cdot u'$
11	Arctangent	$(arctg\ u)' = \frac{1}{1+u^2} \cdot u'$
12	Arccotangent	$(arcctg\ u)' = -\frac{1}{1+u^2} \cdot u'$

(Look at the Appendix E).

The chain rule says that the derivative of composition of functions $f(g(x))$ equals multiplication the derivative of outside function and the derivative inside function

$$\boxed{[f(g(x))]' = f'(g(x)) \cdot g'(x) \text{ or } \frac{d}{dx}[f(g(x))] = \frac{d}{dx}[f(g(x))] \cdot \frac{d}{dx}[g(x)]}$$

Example 5.3 Differentiated functions

a) $y = 5^{ctg 2x} - \sqrt{\ln(3-x^2)}$; b) $y = \sqrt[5]{x} \arccos 4x$.

Solution. a) $y = 5^{ctg 2x} - \sqrt{\ln(3-x^2)}$,

$$y' = \left(5^{ctg 2x}\right)' - \left(\sqrt{\ln(3-x^2)}\right)' = 5^{ctg 2x} \ln 5 \cdot (ctg 2x)' - \frac{\left(\ln(3-x^2)\right)'}{2\sqrt{\ln(3-x^2)}} =$$

$$= 5^{ctg 2x} \ln 5 \cdot \left(\frac{-2}{\sin^2 2x}\right) - \frac{1}{2\sqrt{\ln(3-x^2)}} \cdot \frac{-2x}{3-x^2};$$

б) $y = \sqrt[5]{x} \arccos 4x$,

$$y' = \left(x^{\frac{1}{5}}\right)' \arccos 4x + \sqrt[5]{x}(\arccos 4x)' = \frac{1}{5}x^{\frac{-4}{5}} \arccos 4x + \sqrt[5]{x}\frac{4}{\sqrt{1-16x^2}}.$$

Go on and study topic about the non-common function derivative.

Questions which we have to consider, as

1. How we must differentiate the exponential-power function, or a function in degree function.

2. How we must differentiate the parametric function

3. What is the implicit function and how we must differentiate it.

First, you should understand the exponential-power function (we often call her as a function in degree function), what is it? It is a function which has the form as $y = [f(x)]^{\varphi(x)}$, where $f(x)$ and $\varphi(x)$ - functions. If we need to find her derivative we can use two different techniques, for example,

1) the first technique could be named as *logarithmic differentiation*, because it is performed by two steps. At beginning, we logarithm the function, and then differentiate it.

Remark 5.2: logarithmic function, remember the following formulas

$$\ln(u \cdot v) = \ln u + \ln v\,; \ \ln(\frac{u}{v}) = \ln u - \ln v\,;$$

$$\ln(u)^k = k \cdot \ln u\,,\ k = const\,.$$

Remark 5.3 The same technique we could use if your function presented as $y = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot \ldots$;

2) the second technique involves the formula using

$$y' = \left([f(x)]^{\varphi(x)}\right)' = \left[\varphi'(x) \cdot \ln f(x) + \varphi(x) \cdot \frac{f'(x)}{f(x)}\right] \cdot [f(x)]^{\varphi(x)}.$$

Example 5.4 Find the derivative of the functions

$$\text{a) } y = tgx^{\sqrt{x}}\,,\ \text{b) } y = \frac{(4x+1)^3\ e^{5x^3}}{\log_5(x+4)}$$

Solution. We will use the first technique

a) $y = tgx^{\sqrt{x}}$, $\ln y = \ln\left(tgx^{\sqrt{x}}\right)$, $\ln y = \sqrt{x} \cdot \ln(tgx)$,

$$(\ln y)' = \left(\sqrt{x} \cdot \ln(tgx)\right)', \quad \frac{y'}{y} = \frac{\ln(tgx)}{2\sqrt{x}} + \frac{\sqrt{x}}{tgx \cdot \cos^2 x},$$

$$y' = \left(\frac{\ln(tgx)}{2\sqrt{x}} + \frac{\sqrt{x}}{tgx \cdot \cos^2 x}\right) \cdot y, \quad y' = \left(\frac{\ln(tgx)}{2\sqrt{x}} + \frac{\sqrt{x}}{tgx \cdot \cos^2 x}\right) \cdot tgx^{\sqrt{x}};$$

b) $y = \dfrac{(4x+1)^3\ e^{5x^3}}{\log_5(x+4)}$, $\ln y = \ln \dfrac{(4x+1)^3\ e^{5x^3}}{\log_5(x+4)}$,

$$\ln y = \ln(4x+1)^3 + \ln e^{5x^3} - \ln \log_5(x+4),$$

$$\ln y = 3\ln(4x+1) + 5x^3 - \ln \log_5(x+4),$$

$$\frac{y'}{y} = \frac{12}{4x+1} + 15x^2 - \frac{1}{\log_5(x+4)} \cdot \frac{1}{(x+4)\ln 5},$$

$$y' = \left(\frac{12}{4x+1} + 15x^2 - \frac{1}{\log_5(x+4)} \cdot \frac{1}{(x+4)\ln 5}\right)\frac{(4x+1)^3\ e^{5x^3}}{\log_5(x+4)}.$$

We quite often can meet in engineering the function called as parametric. It is the function as the form $x = \varphi(t)$, $y = \psi(t)$, where t is a parameter. If these function $\varphi(t)$ and $\psi(t)$ are differentiable at the some interval and also the function $\psi(t)$ has inverse function and wherein it is not equal zero ($\varphi'_t(t) \neq 0$) then the derivative of this function equal ratio of the derivation of functions

$$y'_x = \psi'(t) / \varphi'(t) = y'_t / x'_t$$

Example 5.5 Find the derivative of the function

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}.$$

Solution. We can find the derivative of each function separately as x'_t and y'_t: $x'_t = a(1-\cos t); y'_t = a\sin t$. After this we can substitute them in the formula, and we obtain answer:

$$y'_x = \frac{a\sin t}{a(1-\cos t)} = \frac{\sin t}{1-\cos t} = ctg\frac{t}{2}.$$

The process of differentiation of the implicit function is no less interesting than previous themes. The function will be called the implicit function if it has a such form $F(x,y)=0$ or $F(x,y)=\Phi(x,y)$. Then if we need to get its derivative you should differentiate both sides of the function and it should be taken into account that the function y is a complex (composed) function.

Example 5.6 Find the derivative of the function

$$tg\,(2x-y)=x^2y$$

Solution. $tg\,(2x-y)=x^2y$, $tg\,(2x-y)-x^2y=0$,

$$(tg\,(2x-y))' - (x^2y)' = (0)',$$

$$\frac{(2x-y)'}{\cos^2(2x-y)} - (x^2)'y - x^2y' = 0,$$

$$\frac{2-y'}{\cos^2(2x-y)} - 2xy - x^2y' = 0,$$

$$\frac{2}{\cos^2(2x-y)} - \frac{y'}{\cos^2(2x-y)} - 2xy - x^2y' = 0$$

$$\frac{y'}{\cos^2(2x-y)}+x^2y'=\frac{2}{\cos^2(2x-y)}-2xy\ ,$$

$$y'\left(\frac{1}{\cos^2(2x-y)}+x^2\right)=\frac{2}{\cos^2(2x-y)}-2xy\ ,$$

$$y'\left(\frac{1+x^2\cos^2(2x-y)}{\cos^2(2x-y)}\right)=\frac{2-2xy\cos^2(2x-y)}{\cos^2(2x-y)}\ ,$$

$$y'\left(\frac{1+x^2\cos^2(2x-y)}{\cos^2(2x-y)}\right)=\frac{2-2xy\cos^2(2x-y)}{\cos^2(2x-y)}\ ,$$

$$y'=\frac{2-2xy\cos^2(2x-y)}{1+x^2\cos^2(2x-y)}\ .$$

The ***second-order derivative*** or the ***second derivative*** of a function $y=f(x)$ is the derivative of the derivative $f'(x)$. The second derivative is denoted by y'', y''_{xx}, $\frac{d^2y}{dx^2}$, $f''(x)$.

The derivative of the second derivative of a function $y=f(x)$ is called the ***third-order derivative***, $y'''=(y'')'$. The ***n-th-order derivative*** of the function $y=f(x)$ is defined as the derivative of its $(n-1)$th derivative:

$$y^{(n)}=(y^{(n-1)})'.$$

The n-th-order derivative is also denoted by $y_x^{(n)}$, $\frac{d^ny}{dx^n}$, $f^{(n)}(x)$.

When finding higher order derivatives of an implicit function use the same rules as for the finding the first order derivative of an implicit function.

If the function is parametrically defined, then the derivatives of the second order and above are found by the formulas:

$$y''_{xx} = \frac{\left(y'_x\right)'_t}{x'_t}, \; y'''_{xxx} = \frac{\left(y''_{xx}\right)'_t}{x'_t}, \; \ldots, \; y_x^{(n)} = \frac{\left(y_x^{(n-1)}\right)'_t}{x'_t}.$$

Example 5.7 Find the third-order derivative of the function:

$$y = \frac{1}{6}\left(x^2+9\right)\left(x-6\right).$$

Solution:

$$y' = \left(\frac{1}{6}\left(x^2+9\right)\left(x-6\right)\right)' = \frac{1}{6}\cdot\left(\left(x^2+9\right)'\cdot\left(x-6\right)+\left(x^2+9\right)\cdot\left(x-6\right)'\right) =$$

$$= \frac{1}{6}\cdot\left(2x\cdot\left(x-6\right)+\left(x^2+9\right)\cdot 1\right) = \frac{1}{6}\cdot\left(2x^2-12x+x^2+9\right) = \frac{1}{6}\left(3x^2-12x+9\right),$$

$$y'' = \left(\frac{1}{6}\left(3x^2-12x+9\right)\right)' = \frac{1}{6}\cdot\left(6x-12\right) = \frac{1}{6}\cdot 6\cdot\left(x-2\right) = x-2,$$

$$y''' = \left(x-2\right)' = 1.$$

Lecture 6 APPLICATION OF THE DIFFEFERENTIAL CALCULUS

L'Hospital's rule. If $f(x)$ and $g(x)$ are both infinitesimals or both infinites as $x \to a$, that is, if the quotient $\dfrac{f(x)}{g(x)}$, at $x = a$, is one of the indeterminate forms $\left[\dfrac{0}{0}\right]$ or $\left[\dfrac{\infty}{\infty}\right]$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

provided that the limit of the ratio of derivatives exists.

The rule is also applicable when $a = \infty$.

If the quotient $\dfrac{f'(x)}{g'(x)}$ again yields an indeterminate form, at the point $x = a$, of one of the two above-mentioned types and $f'(x)$ and $g'(x)$ satisfy all the requirements that have been stated for $f(x)$ and $g(x)$, we can then pass to the ratio of second derivatives, etc.

Example 6.1 Compute $\lim\limits_{x \to 0}\left(\dfrac{1}{\sin^2 x} - \dfrac{1}{x^2}\right)$.

Solution:

$$\lim_{x \to 0}\left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right) = \frac{1}{\sin^2 0} - \frac{1}{0^2} = \frac{1}{0} - \frac{1}{0} = [\infty - \infty] =$$

Reducing to a common denominator, we get

$$= \lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \frac{0^2 - \sin^2 0}{0^2 \cdot \sin^2 0} = \left[\frac{0}{0}\right] =$$

Before applying the L'Hospital's rule, we will use one of special limits for trigonometric functions, i.e., $\lim\limits_{\alpha\to 0}\dfrac{\alpha}{\sin\alpha}=1$:

$$=\lim_{x\to 0}\frac{\left(x^2-\sin^2 x\right)\cdot x^2}{x^2\sin^2 x\cdot x^2}=\lim_{x\to 0}\frac{x^2}{\sin^2 x}\cdot\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\cdot x^2}=1\cdot\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^4}=$$

The L'Hospital's rule gives

$$=\lim_{x\to 0}\frac{\left(x^2-\sin^2 x\right)'}{\left(x^4\right)'}=\lim_{x\to 0}\frac{2x-2\sin x\cos x}{4x^3}=\lim_{x\to 0}\frac{2x-\sin 2x}{4x^3}=$$

$$=\frac{2\cdot 0-\sin 0}{4\cdot 0^3}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\left(2x-\sin 2x\right)'}{\left(4x^3\right)'}=\lim_{x\to 0}\frac{2-2\cos 2x}{12x^2}=$$

$$=\lim_{x\to 0}\frac{1-\cos 2x}{6x^2}=\lim_{x\to 0}\frac{2\sin^2 x}{6x\cdot x}=\lim_{x\to 0}\frac{\sin x\cdot\sin x}{3\cdot x\cdot x}=\frac{1}{3}\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{x}=\frac{1}{3}.$$

To evaluate an indeterminate form like $[0\cdot\infty]$, one should transform the appropriate product $f(x)\cdot g(x)$, into the quotient $\dfrac{f(x)}{\frac{1}{g(x)}}$ or $\dfrac{g(x)}{\frac{1}{f(x)}}$ to get one of the indeterminate forms $\left[\dfrac{0}{0}\right]$ or $\left[\dfrac{\infty}{\infty}\right]$, and then to apply the L'Hospital's rule.

Example 6.2 Compute $\lim\limits_{x\to 0}x^2\cdot\ln x$.

Solution:

$$\lim_{x\to 0}x^2\cdot\ln x=0^2\cdot\ln 0=[0\cdot\infty]=\lim_{x\to 0}\frac{\ln x}{\frac{1}{x^2}}=\frac{\ln 0}{\frac{1}{0^2}}=\left[\frac{\infty}{\infty}\right]=$$

$$= \lim_{x\to 0}\frac{(\ln x)'}{\left(x^{-2}\right)'} = \lim_{x\to 0}\frac{\frac{1}{x}}{-2x^{-3}} = \lim_{x\to 0}\frac{x^3}{-2x} = \lim_{x\to 0}\frac{x^2}{-2} = \frac{0^2}{-2} = 0 .$$

Example 6.3 Compute $\lim_{x\to+\infty}\left(e^x + x\right)^{\frac{1}{x}}$.

Solution:

$$\lim_{x\to+\infty}\left(e^x + x\right)^{\frac{1}{x}} = \left(e^{+\infty} + \infty\right)^{\frac{1}{+\infty}} = \left[\infty^0\right].$$

Taking logarithms and applying the L'Hospital's rule, we get

$$\lim_{x\to 0}\ln\left(e^x + x\right)^{\frac{1}{x}} = \lim_{x\to 0}\frac{1}{x}\cdot\ln\left(e^x + x\right) = \lim_{x\to 0}\frac{\ln\left(e^x + x\right)}{x} = \left[\frac{\infty}{\infty}\right] =$$

$$= \lim_{x\to\infty}\frac{\left(\ln\left(e^x + x\right)\right)'}{x'} = \lim_{x\to\infty}\frac{\frac{e^x+1}{e^x+x}}{1} = \lim_{x\to\infty}\frac{e^x+1}{e^x+x} = \frac{e^\infty+1}{e^\infty+\infty} = \left[\frac{\infty}{\infty}\right] =$$

$$= \lim_{x\to\infty}\frac{\left(e^x+1\right)'}{\left(e^x+x\right)'} = \lim_{x\to\infty}\frac{e^x}{e^x+1} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty}\frac{\left(e^x\right)'}{\left(e^x+1\right)'} = \lim_{x\to\infty}\frac{e^x}{e^x} = 1 .$$

Therefore, $\lim_{x\to+\infty}\left(e^x + x\right)^{\frac{1}{x}} = e$.

Now we should consider the topic about the behavior of the function and how to determine it with the function derivative concept, namely, search answer about the function decreasing and increasing.

Increasing or decreasing?

Let $y = f(x)$ be continuous on an interval X and differentiable on the interior of X

1) If $f'(x) > 0$ for all $x \in X$, then function *is increasing on* X

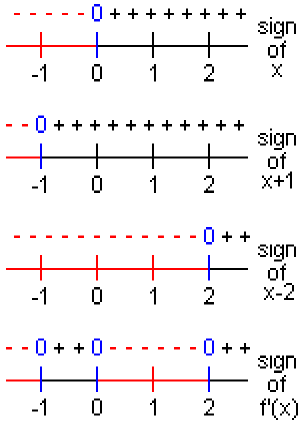
2) If $f'(x) < 0$ for all $x \in X$, then function *is decreasing on* X

Example 6.4 The function $y = 3x^4 - 4x^3 - 12x^2 + 3$ has the first derivative as

$$y' = 12x^3 - 12x^2 - 24x =$$

$$= 12x(x^2 - x - 2) = 12x(x-2)(x+1).$$

If we trace the sign of the derivative, we will see that it changes (pictures on the left). Thus, $f(x)$ is increasing on intervals $(-1,0) \cup (2,\infty)$ and decreasing on $(-\infty,1) \cup (0,2)$.



Definition 6.1 Points in which the derivative equals zero or not exists we will call *critical points (critical number)*.

To know how to find the extremum of a function (the highest or lowest point on the interval where the function is defined) we should calculate the first derivative of the function and make a study of sign. The extremum of a function is reached when this derivative is equal to zero and changes of sign.

Definition 6.2 A minimum of a function m exists for all x if $f(x) \geq m$ is greater than or equal to a minimum.

Definition 6.3 A maximum of a function M exists for all x if $f(x) \leq M$ is less than or equal to a maximum.

An extremum of a function is always defined over interval (that may be domain of definition of a function). For example, the function $f(x) = x^2$ defined over R (that $x \in (-\infty;+\infty)$), it has minimum in $x = 0$ because $f(x) \geq 0$ over R (for all $x \neq 0$).

Suppose the function in question is continuous and differentiable in the interval. Then, there are a few shortcuts to determining extrema. All local extrema are points at which the derivative is zero (though it is possible for the derivative to be zero and for the point not to be a local extrema). While they can still be endpoints (depending upon the interval in question), the absolute extrema may be determined with a few shortcuts too. These are the derivative tests.

Theorem 6.1 ***(The First Derivative Test)***

Suppose $y = f(x)$ is a real-valued function and it has an interval on which it is defined and differentiable. Then, if x_0 is a critical point of $y = f(x)$ in,

1. If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0 then a function has a relative maximum at x_0.

2. If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0 then a function has a relative minimum at x_0 .

3. If $f'(x)$ has the same sign on both open interval extending left from x_0 and an open interval extending right from x_0 then a function does not have a relative extremum at x_0.

In simpler terms, a point is a maximum of a function if the function increases before and decreases after it. Conversely, a point is a minimum if the function *decreases* before and increases after it.

There may not exist an absolute maximum or minimum if the region is unbounded in either the positive or negative direction or if the function is not continuous. If the function is not continuous (but is bounded), there will still exist a supremum or infinum, but there may not necessarily exist absolute extrema. If the function is continuous and bounded and the interval is closed, then there must exist an absolute maximum and an absolute minimum. If a function is not continuous, then it may have absolute extrema at any points of discontinuity. Generally, absolute extrema will only be useful for functions with at most a finite number of points of discontinuity. The absolute extrema can be found by considering these points together with the following method for continuous portions of the function.

All local maximums and minimums on a function's graph — called local extrema — occur at critical points of the function (where the derivative is zero or undefined). (Don't forget, though, that not all critical points are necessarily local extrema.)

The first step in finding a function's local extrema is to find its critical numbers (the x-values of the critical points). You then use the First Derivative Test. This test is based on the Nobel-prize-caliber ideas that as you go over the top of a hill, firstly, you go up and then you go down, and that when you drive into and out of a valley, you go down and then up. This calculus stuff is pretty amazing, eh?

Example 6.5 The figure 6.1 shows the graph of

$$y = 3x^5 - 20x^3$$

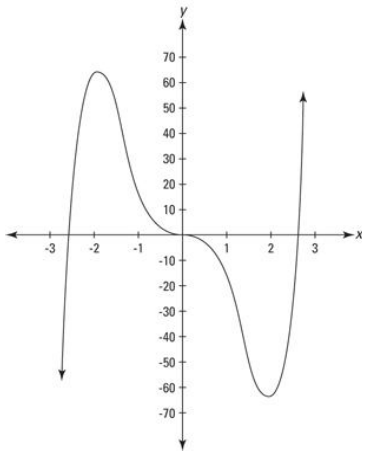


Figure 6.1

Find the critical numbers of this function, here's what you do.

1. Find the first derivative of function using the power rule.

$$y' = \left(3x^5 - 20x^3\right)' = 15x^4 - 60x^2$$

2. Set the derivative equal to zero and solve for x.

$$15x^4 - 60x^2 = 0, 15x^2\left(x^2 - 4\right) = 0,$$

$$15x^2(x-2)(x+2) = 0, \ x = 0 \text{ or }$$

$$x = 2 \text{ or } x = -2.$$

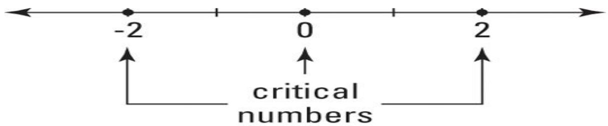
lues are the critical numbers of $y = f(x)$. Additional critical numbers could exist if the first derivative were undefined at some x-values, but because the derivative

$$y' = 15x^4 - 60x^2$$

is defined for all input values, the above solution set, $0;2;-2$, is the complete list of critical numbers. Because the derivative (and the slope) of $y = f(x)$ equals zero at these three critical numbers, the curve has horizontal tangents at these numbers.

Now that you've got the list of critical numbers, you need to determine whether peaks or valleys or neither occur at those x-values. You can do this with the First Derivative Test. Here's how:

1. Take a number line and put down the critical numbers you have found: 0, –2, and 2.



You divide this number line into four regions: to the left of –2, from –2 to 0, from 0 to 2, and to the right of 2.

2. Pick a value from each region, plug it into the first derivative, and note whether your result is positive or negative.

For this example, you can use the numbers –3, –1, 1, and 3 to test the regions.

$$y'(-3) = 15(-3)^4 - 60(-3)^2 = 15 \cdot 81 - 60 \cdot 9 = 675 > 0,$$

$$y'(-1) = 15(-1)^4 - 60(-1)^2 = 15 - 60 = -45 < 0,$$

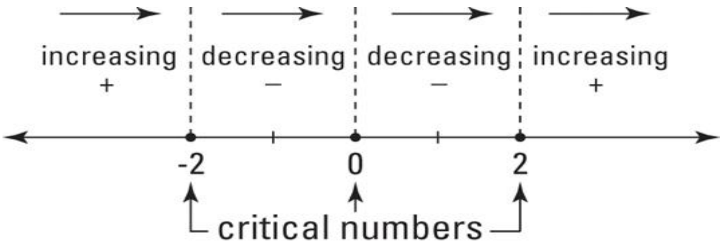
$$y'(3) = 15(3)^4 - 60(3)^2 = 15 \cdot 81 - 60 \cdot 9 = 675 > 0,$$

$$y'(1) = 15(1)^4 - 60(1)^2 = 15 - 60 = -45 < 0$$

These four results are, respectively, positive, negative, negative, and positive.

3. Take your number line, mark each region with the appropriate positive or negative sign, and indicate where the function is increasing and decreasing.

It's increasing where the derivative is positive and decreasing where the derivative is negative. The result is a so-called sign graph for the function.



This figure simply tells you what you already know if you've looked at the graph of *function* that the function goes up until –2, down from –2 to 0, further down from 0 to 2, and up again from 2 on.

Now, here's the rocket science. The function switches from increasing to decreasing at –2; in other words, you go up to –2 and then down. So, at –2, you have a hill or a local maximum. Conversely, because the function switches from decreasing to increasing at 2, you have a valley there or a local minimum. And because the sign of the first derivative doesn't switch at zero, there's neither a min nor a max at that x-value.

4. Obtain the function values (in other words, the heights) of these two local extrema by plugging the x-values into the original function.

$y(-2) = 3(-2)^5 - 20(-2)^3 = 64$, $y(2) = 3(2)^5 - 20(2)^3 = -64$

Thus, the local max is located at (–2, 64), and the local min is at (2, –64). You're done.

Theorem 6.2 *(The Second Derivative Test)*

Suppose $y = f(x)$ is a real-valued function and $x \in X$ is an interval on which $y = f(x)$ is defined and twice-differentiable. Then, if x_0 is a critical point:

1. If $f''(x_0) > 0$, then $y = f(x)$ has a local minimum at the x_0.

2. If $f''(x_0) < 0$, then $y = f(x)$ has a local maximum at the x_0.

Remark 6.1 In simpler terms, a point is a maximum of a function if the function is concave down, and a point is a minimum of a function if the function is concave up.

The derivative tests may be applied to local extrema as well, given a sufficiently small interval. In fact, the second derivative test itself is sufficient to determine whether a potential local extremum (for a differentiable function) is a maximum, a minimum, or neither.

Example 6.6 Find the extrema of a function $f(x) = (x-1)^3 (x+1)^2$

Solution: Our function is defined for $D(x): x \in (-\infty, +\infty)$; $E(y): y \in (-\infty, +\infty)$.

2) Calculate the first derivative:

$$f'(x) = 3(x-1)^2 (x+1)^2 + 2(x+1)(x-1)^3.$$

3) And find the critical point due to solve the equations as $f'(x) = 0$:

$$(x-1)^2 (x+1)(5x+1) = 0, \ (x-1)^2 = 0, \ x+1 = 0, \ 5x+1 = 0,$$

so, we obtained $x_1 = 1, x_2 = -1, x_3 = -1/5$ are critical points.

Exanimate the sign of $f''(x)$ (the second derivative) at the all-out critical points, however, first of all we will calculate the second derivative:

$$f''(x) = 2(x-1)(x+1)(5x+1) + (x-1)^2 (5x+1+5(x+1)) =$$

$$= (x-1)(10x^2 + 12x + 2 + 10x^2 - 4x - 6) =$$

$$(x-1)(20x^2 + 8x - 4);$$

after that we substitute each values of critical points at the expression of the second derivative and we will get the following results as $f''(-1) = -16 < 0$, so at this point we will have a local maximum; $f''(-1/5) = 114/25 > 0$, so at this point we will have a local minimum,

$f''(1)=0$, unfortunately, we can't say definitely, we need to perfume other acts to examine this point. Take appoint $x=0$, which located on the left hand of this point від $x=1$, and define the sign of the first derivative, it is: $f'(0)=1>0$; then we take a point $x=2$, on the right hand of $x=1$, and define the sign of the first derivative, it is: $f'(2)=33>0$. Insomuch as the first derivative don't change the sign nearby of the point $x=1$, so we don't have the extremum at this point. Compute the values of extrema:

$$x_{\min}=-1/5 \Rightarrow y_{\min}=y(-1/5)=-3456/3125,$$

$$x_{\max}=-1 \Rightarrow y_{\max}=y(-1)=0\,.$$

Example 6.7 Find the extrema of the function

$$y=\frac{x}{1+x^2}.$$

Solution. The domain of function definition is:

$$1+x^2\neq 0,\ x\in R.$$

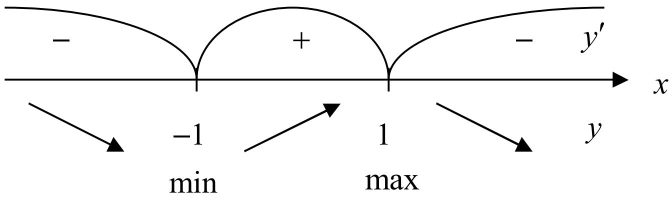
Find the derivative of the given function:

$$y'=\left(\frac{x}{1+x^2}\right)'=\frac{x'\cdot\left(1+x^2\right)-x\cdot\left(1+x^2\right)'}{\left(1+x^2\right)^2}=\frac{1\cdot\left(1+x^2\right)-x\cdot 2x}{\left(1+x^2\right)^2}=\frac{1+x^2-2x^2}{\left(1+x^2\right)^2}=\frac{1-x^2}{\left(1+x^2\right)^2}.$$

Solve the equation $y'=0$:

$$\frac{1-x^2}{\left(1+x^2\right)^2}=0,\ 1-x^2=0,\quad x^2=1,\quad x=\pm 1\,.$$

Put all critical point taking into account the point where our function does not exist (or undetermined) on the axis and investigate the sign at the obtained intervals.



To determine what the sign of y' is in the interval $(-\infty;-1)$, it is sufficient to determine the sign of y' at some point of the interval. For example, taking $x=-2$, we get

$$y'(-2)=\frac{-2}{1+(-2)^2}=\frac{-2}{1+4}=-\frac{2}{5},$$

hence, $y'<0$ in the interval $(-\infty;-1)$ and the function in this interval decreases.

Therefore, the function increases in the interval $x\in(-1,1)$, decreases in the interval $x\in(-\infty,-1)\cup(1,+\infty)$, $x=-1$ is the minimum point of the function, $x=1$ is the maximum one.

$$y_{\min}(-1)=\frac{-1}{1+(-1)^2}=-0,5,\quad y_{\max}(1)=\frac{1}{1+1^2}=0,5,$$

$A(-1;-0,5)$, $B(1;0,5)$ are the extremal points.

Find maximum and minimum values of a function over a closer interval.

Let a function $f(x)$ be a function on $[a,b]$ and c is a inner point in the interval $[a,b]$. Then:

1) if for any point x in $[a,b]$, $f(x)\ge f(c)$ (respectively, $f(x)\le f(c)$) then $f(c)$ is the absolute (or global) minimum value (respectively, absolute (or global) maximum value) of $f(x)$ on $[a,b]$;

2) if $a<c<b$, and for any point x in an open interval containing c, $f(x)\geq f(c)$ (respectively, $f(x)\leq f(c)$) then $f(c)$ is the local minimum value $f(x)$ (respectively, local maximum value) on $[a,b]$:

3) if $f(x)$ is continuous on $[a,b]$ and differentiable in (a,b), a point c in $[a,b]$ is a critical point of a function $f(x)$ if either $f'(c)$ does not exist, or $f'(c)=0$;

4) important:_ if $f(x)$ is continuous on $[a,b]$ and differentiable in (a,b) and if for some c in (a,b), $f(c)$ is the local minimum or maximum, then c must be a critical point. Any absolute minimum or maximum must take place at critical point inside the interval or at the boundaries point a and b.

Example 6.8 Find the largest and smallest values of a function $y=x^3+9x^2-1$ on a segment $[-2,2]$.

Solution. Since

$$y'=\left(x^3+9x^2-1\right)'=3x^2+18x,$$

it follows that the critical points of the function are

$$3x^2+18x=0,$$

$$3x\cdot\left(x+6\right)=0,$$

$$x=0\in\left[-2;2\right],\quad x=-6\notin\left[-2;2\right].$$

Comparing the values of the function at $x=0$ and at the endpoints of the given interval

$$y\left(0\right)=0^3+9\cdot 0^2-1=-1,$$

$$y\left(-2\right)=\left(-2\right)^3+9\cdot\left(-2\right)^2-1=-8+36-1=27,$$

$$y(2) = 2^3 + 9 \cdot 2^2 - 1 = 8 + 36 - 1 = 43,$$

we conclude that the function attains its smallest value $m = -1$ at the point $x = 0$ and the greatest value $M = 43$ at the point $x = 2$:

$$\max_{[-2;2]} y = y(2) = 43, \quad \min_{[-2;2]} y = y(0) = -1.$$

Example 6.9 Find the maximum value and the minimum value attained by $f(x) = \dfrac{1}{x(1-x)}$ in the interval $[2,3]$.

Solution. Note that the domain of $f(x)$ does not contain $x = 0$ and $x = 1$, and these points are not in the interval $[2,3]$. Find critical points. Compute

$$f'(x) = -\frac{1-2x}{x^2(1-x)^2}, \; f'(x) = 0, \; -\frac{1-2x}{x^2(1-x)^2} = 0, \; 2x = 1, \; x = \frac{1}{2}.$$

Therefore, the only possible critical point is $x = \dfrac{1}{2}$. As this point is not in the interval $[2,3]$, it is not a critical point. Compute $f(x)$ only at the boundaries of the closed interval

$$f(3) = \frac{1}{3(1-3)} = -\frac{1}{6}, \; f(2) = \frac{1}{2(1-2)} = -\frac{1}{2}.$$

Compare the data resulted in Step 2 to make conclusions: $f(x)$ attains its absolute maximum value $f(3) = -\dfrac{1}{6}$ at $x = 3$ and f(x) attains its absolute minimum value $f(2) = -\dfrac{1}{2}$ at $x = 2$.

Lecture 7 THE INDEFINITE INTEGRAL. METHODS OF INTEGRATION.

Anti-differentiation or integration is the reverse process to differentiation. For example, if $f(x)=x^2$, we know that this is the derivative of $F(x)=\frac{x^3}{3}$. Could there be any other possible answers? If we shift the cube parabola $F(x)=\frac{x^3}{3}$ by sliding it up or down vertically, all the points on the curve will still have the same tangent slopes, i.e. derivatives.

Definition 7.1 Where possible, check your answer by differentiating, remembering that the derivative of a constant, C, is zero. In mathematical notation, this anti-derivative is written as

$$\int f(x)dx = F(x)+C\text{, where } F'(x)=f(x).$$

In words, if the derivative of $F(x)$ is $f(x)$, then we say that *an indefinite integral* of $f(x)$ with respect to x is $F(x)$. The integration symbol "$\int$ " is an extended S for "summation". the "dx" part indicates that the integration is with respect to x. For instance, the integral $\int x^2 dt$ cannot be found, unless x can be rewritten as some function of t as $\int t^2 dt$.

Remark 7.1 However, you are NOT encouraged to memorize integration formulae, but rather to become VERY familiar with the list of derivatives and to practice recognizing a function as the derivative of another function. If you try memorizing both differentiation and integration formulae, you will one day mix them up and use the wrong one. And there is absolutely no need to memorize the integration formulae if you know the differentiation ones. It is much better to recall the way in which an integral is defined as an anti-derivative. Every time you perform integration you should pause for a moment and check it by differentiating the answer to see if you get back the function you began with. This is a very important habit to develop. There is no need to write down the checking process every time, often you will do it in your head, but if you get into this habit, you will avoid a lot of mistakes

Table 7.1 – The standard integrals

Basic integrals			
1	$\int 0\, du = C$	5	$\int \sin u\; du = -\cos u + C$
2	$\int u^{\alpha}\, du = \dfrac{u^{\alpha+1}}{\alpha+1} + C$	6	$\int \cos u\; du = \sin u + C$
2a	$\int du = u + C$	7	$\int \dfrac{du}{\cos^2 u} = tg\; u + C$
2б	$\int \dfrac{du}{\sqrt{u}} = 2\sqrt{u} + C$	8	$\int \dfrac{du}{\sin^2 u} = -\; ctg\; u + C$
2в	$\int \dfrac{du}{u^2} = -\dfrac{1}{u} + C$	9	$\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin \dfrac{u}{a} + C$
3	$\int \dfrac{du}{u} = \ln\lvert u\rvert + C$	10	$\int \dfrac{du}{\sqrt{u^2 + b}} = \ln\left\lvert u + \sqrt{u^2 + b}\right\rvert + C$
4	$\int a^u\, du = \dfrac{a^u}{\ln a} + C$	11	$\int \dfrac{du}{u^2 + a^2} = \dfrac{1}{a}\, arctg \dfrac{u}{a} + C$
4a	$\int e^u\, du = e^u + C$	12	$\int \dfrac{du}{u^2 - a^2} = \dfrac{1}{2a} \ln \left\lvert \dfrac{u - a}{u + a} \right\rvert + C$

Some rules for calculating integrals. Properties.

Rule 1. $\int C \cdot f(x)dx = C\int f(x)dx$

Rule 2.

$$\int \left[f_1(x) + f_2(x) - f_3(x) \right] dx = \int f_1(x)dx + \int f_2(x)dx + \int f_3(x)dx$$

Rule 3. $\left(\int f(x)dx \right)' = f(x)$

Rule 4. $d\int f(x)dx = f(x)dx$

Rule 5. $\int dF(x) = F(x) + C$

The most interesting rules for using will be rules 1 and 2. Presented it.

Example 7.1 To find the indentified integrals:

$$\text{a) } \int\left(x^3+\sqrt[4]{x}-\frac{2}{\sqrt[5]{x^3}}\right)dx\,,\ \text{b) } \int\frac{dx}{\sqrt{7-9x^2}}\,.$$

Solution: a) using rules 1,2 and according to the standard integral (basic integral formula №2), we will get:

$$\int\left(x^3+\sqrt[4]{x}-\frac{2}{\sqrt[5]{x^3}}\right)dx=\int x^3dx+\int\sqrt[4]{x}dx-\int\frac{2}{\sqrt[5]{x^3}}dx=\int x^3dx+\int x^{1/4}dx-$$

$$-2\int x^{-3/5}dx=\frac{x^4}{4}+\frac{x^{1/4+1}}{1/4+1}-2\frac{x^{-3/5+1}}{-3/5+1}+C=\frac{x^4}{4}+\frac{4x^{5/4}}{5}-\frac{10x^{2/5}}{2}+C\,;$$

b) using formula 9 for basic integrals, we have performed some transformations before, we will get:

$$\int\frac{dx}{\sqrt{7-9x^2}}=\int\frac{dx}{\sqrt{9\left(\frac{7}{9}-x^2\right)}}=\frac{1}{3}\int\frac{dx}{\sqrt{\left(\frac{7}{9}-x^2\right)}}=\frac{1}{3}\arcsin\frac{3x}{\sqrt{7}}+C\,.$$

Consider some useful methods for the non-table integrals.

Integration by substitution

Integration by Substitution (also called "u-Substitution" or "The Reverse Chain Rule") is a method to find an integral, but only when it can be set up in a special way. The first and most vital step is to be able to write our integral in this form:

$$\int f(\varphi(x))\cdot\varphi'(x)dx=\left|\begin{array}{c}t=\varphi(x),\\ dt=\varphi'(x)dt\end{array}\right|=\int f(t)dt=F(t)+C=F(\varphi(x))+C$$

or

$$\int f(x)dx = \left| \begin{matrix} x=\varphi(t); \\ dx=\varphi'(t)dt \end{matrix} \right| = \int f(\varphi(t))\varphi'(t)dt = H(t)+C == \left| t=\varphi^{-1}(x) \right| = H(\varphi^{-1}(x))+C .$$

Example 7.2 To find the indentified integral: $\int \frac{e^{tg2x}}{\cos^2 2x}dx$.

Solution.

$$\int \frac{e^{tg2x}}{\cos^2 2x}dx = \int e^{tg2x}\frac{1}{\cos^2 2x}dx = \left| \begin{matrix} t=tg2x, \\ dt=\frac{2}{\cos^2 2x}dx \Big| :2, \\ \frac{dt}{2}=\frac{1}{\cos^2 2x}dx \end{matrix} \right| =$$

$$= \int e^t \frac{dx}{2} == \frac{1}{2}\int e^t dt == \frac{1}{2}e^t + C = \frac{1}{2}e^{tg2x} + C .$$

Remark 7.2 We must come back to the $\varphi(x)$.

Integration by parts

Consider the integral of product of function as $\int f(x)\cdot g(x)dx$. To do this integral we will need to use integration by parts so let's derive the integration by parts formula. We'll start with the product rule.

Definition 7.2 Let $u=u(x)$ and $v=v(x)$ are two continuous functions, they have continuous derivatives. As we know, the differential of products of those functions is

$$d(u\cdot v)=vdu+udv .$$

Now, integrate both sides of this.

$$\int d(uv)=\int vdu+\int udv ,$$

or

$$\boxed{\int u\,dv = uv - \int v\,du}.$$

The last formula is *formula integration by parts*.

Remark 7.3 To use this formula, we will need to identify *u* and *dv*, compute *du* and *v* and then use the formula. Note as well that computing *v* is very easy. All we need to do is integrate *d:* $v = \int dv$.

Example 7.3 Evaluate the following integrals: $\int x \cdot 2^{3x}\,dx$

Solution. Notice as well that in doing integration by parts anything that we choose for u will be differentiated. So, it seems that choosing $u = x$ will be a good choice since upon differentiating the x will drop out. Now that we've chosen u we know that dv will be everything else that remains. So, here are the choices for u and dv as well as du and v.

$$\int x \cdot 2^{3x}\,dx = \left| \begin{array}{l} u = x, \\ dv = 2^{3x}\,dx, \\ du = dx, \\ v = \int 2^{3x}\,dx = \dfrac{1}{3} \cdot \dfrac{2^{3x}}{\ln 2} \end{array} \right| = uv - \int v\,du = x \cdot \frac{1}{3} \cdot \frac{2^{3x}}{\ln 2} -$$

$$- \frac{1}{3\ln 2} \int 2^{3x}\,dx = \frac{x \cdot 2^{3x}}{3\ln 2} - \frac{1}{3\ln 2} \cdot \frac{2^{3x}}{3\ln 2} + C.$$

However, those methods are not universal, so sometimes we should use appropriated techniques to some functions classes' integrals. Consider any of them that are most common.

Integration of rational fractions

We will base on two theorems:

Theorem 7.1 (*Fundamental Theorem of Algebra over the Real Numbers*). A real polynomial of degree $n \geq 1$ can be factored as a constant times a product of linear factors $x - x_1$ and irreducible quadratic factors $ax^2 + bx + c$.

Note that $ax^2 + bx + c = a(x - x_1)(x - x_2)$, where $x_1 = \alpha + i\beta$, $x_1 = \alpha - i\beta$ are complex conjugates.

Theorem 7.2 Every rational function $\frac{P_m(x)}{Q_n(x)}$ when degree of $P_m(x)$ less than degree of $Q_n(x)$, $n \geq m$, can be decomposed into partial fraction.

According to that: the rational function $\frac{P_m(x)}{Q_n(x)}$, where $P_m(x)$ на $Q_n(x)$ are both polynomials, can be integrated in four steps:

4. Reduce the fraction if it is improper (i.e. degree of $P_m(x)$ is greater than degree of $Q_n(x)$;
5. Factor $Q_n(x)$ into linear and/or quadratic (irreducible) factors;
6. Decompose the fraction into a sum of partial fractions;
7. Calculate integrals of each partial fraction.

Consider these steps in more details.

Step 1 Reducing an Improper Fraction

If the fraction is improper (i.e. degree of $P_m(x)$ is greater than degree of $Q_n(x)$), divide the numerator $P_m(x)$ by the denominator $Q_n(x)$ to obtain $\frac{P_m(x)}{Q_n(x)} = G_{m-n}(x) + \frac{R_k(x)}{Q_n(x)}$, where $\frac{R_k(x)}{Q_n(x)}$ is a proper fraction.

Step 2 Factoring $Q_n(x)$ into Linear and/or Quadratic Factors

Write the denominator $Q_n(x)$ as

$$Q_n(x) = (x-a)^k \cdot \ldots \cdot (x^2 + px + q)^t \cdot \ldots,$$

where quadratic functions are irreducible, i.e. do not have real roots.

Step 3 Decomposing the Rational Fraction into a Sum of Partial Fractions.

Write the function as follows:

$$\frac{P(x)}{Q(x)}=\frac{A_k}{(x-a)^k}+\frac{A_{k-1}}{(x-a)^{k-1}}+\frac{A_{k-2}}{(x-a)^{k-2}}+\ldots+\frac{A_1}{(x-a)}+$$

$$\frac{A_t x+B_t}{\left(x^2+px+q\right)^t}+\frac{A_{t-1}x+B_{t-1}}{\left(x^2+px+q\right)^{t-1}}+\ldots+\frac{A_1 x+B_1}{\left(x^2+px+q\right)}+\ldots$$

The total number of undetermined coefficients A_k, $A_t, B_t, \ldots$ must be equal to the degree of the denominator $Q_n(x)$. Then equate the coefficients of equal powers of x by multiplying both sides of the latter expression by $Q_n(x)$ and write the system of linear equations in A_k, $A_t, B_t, \ldots$ The resulting system must always have a unique solution.

Step 4 Integrating partial fractions.

Use the following formulas to evaluate integrals of partial fractions with linear and quadratic denominators:

1) $\int \frac{Adx}{x-a} = A\ln|x-a|+C$;

2) $\int \frac{Adx}{(x-a)^k} = A\int (x-a)^{-k}dx = A\frac{(x-a)^{-k+1}}{-k+1}+C$, $k \geq 2$;

3) $\int \frac{Ax+B}{x^2+px+q}dx$;

4) $\int \frac{Ax+B}{x^2+px+q}dx$.

Consider some cases of integration of rational function.

CASE 1. Distinct linear factors.

Example 7.4 Evaluate the following integral:

$$I=\int \frac{4x^2-13x+7}{(x^2-5x+6)(x+1)}dx \cdot$$

Solution. $I=\int \frac{4x^2-13x+7}{(x-2)(x-3)(x+1)}dx=\int \frac{A\,dx}{x-2}+\int \frac{B\,dx}{x-3}+\int \frac{C\,dx}{x+1}$.

There

$4x^2 - 13x + 7 = A(x-3)(x+1) + B(x-2)(x+1) + C(x-2)(x-3)$.

Find the coefficients (do it by yourself)), obtain: $A = 1$; $B = 1$; $C = 2$.

$$I = \int \frac{dx}{x-2} + \int \frac{dx}{x-3} + \int \frac{2dx}{x+1} = \ln|x-2| + \ln|x-3| + 2\ln|x+1| + C.$$

CASE 2. Repeated linear factors

Example 7.5 Compute $\int \frac{4x^2 + 5x + 8}{(x+2)^2 (x-1)} dx.$

Solution. We have two repeated factors, so our partial fractions will have the form as

$$\int \frac{4x^2 + 5x + 8}{(x+2)^2 (x-1)} dx = \int \left(\frac{A}{(x+2)^2} + \frac{B}{(x+2)} + \frac{C}{x-1} \right) dx =$$

we reduce them to the common denominator and obtain

$$\int \frac{A(x-1) + B(x+2)(x-1) + C(x+2)^2}{(x-1)(x+2)^2} dx =$$

We will use the method of dominate roots to find the undetermined coefficients. Since the number 2 of multiple roots, then the third value of the variable x will be chosen arbitrarily

$$= \left| \begin{array}{c} A(x-1) + B(x+2)(x-1) + C(x+2)^2 = 4x^2 + 5x + 8, \\ (x-1)(x+2)^2 = 0, \ x = 1, \ x = -2, \ x = -1, \\ x = 1, \ 9C = 17, \ C = \dfrac{17}{9}, \\ x = -2, \ -3A = 14, \ A = \dfrac{-14}{3}, \\ x = -1, \ -2A - 2B + C = 9, \ -2B = \dfrac{-20}{9}, B = \dfrac{10}{9} \end{array} \right| =$$

$$=\int\left(\frac{-14/3}{(x+2)^2}+\frac{10/9}{(x+2)}+\frac{17/9}{x-1}\right)dx=\frac{14}{3(x+2)}+\frac{10}{9}\ln|x+2|+\frac{17}{9}\ln|x-1|+C.$$

Example 7.6 Evaluate the following integral

$$\int\frac{-2x^4-x^3-6x^2+18x+13}{x^5-x^4-3x^3-5x^2-10x-6}dx.$$

Solution. $P(x)=-2x^4-x^3-6x^2+18x+13$,

$Q(x)=x^5-x^4-3x^3-5x^2-10x-6$.

Polynomial $Q(x)$ can be factorized as follows: $Q(x)=(x+1)^2(x-3)(x^2+2)$, then the required expansion has the form:

$$\frac{P(x)}{Q(x)}=\frac{A}{x-3}+\frac{B}{(x+1)^2}+\frac{C}{x+1}+\frac{Dx+E}{x^2+2},$$

where numbers A,B,C,D i E we should find. We reduce the right-hand side to the common denominator, that is $Q(x)$, according to the condition of equality of fractions, we will get identity for polynomials:

$$\begin{aligned}&A(x+1)^2(x^2+2)+B(x-3)(x^2+2)+C(x-3)(x+1)(x^2+2)\\&=+(Dx+E)(x-3)(x+1)^2=P(x)\end{aligned}.$$

Find unknown coefficients A,B,C,D,E by the method of undetermined coefficients A,B,C,D,E. We open parentheses and give similar, equate the coefficients at the same powers of x in the left and right sides of our identity. We obtain a system of five equations with five unknowns and solve its Gauss method:

$$\begin{array}{c|l}x^4 & A+C+D=-2,\\ x^3 & 2A+B-2C-D+E=-1,\\ x^2 & 3A-3B-C-5D-E=-6,\\ x^1 & 4A+2B-4C-3D-5E=18,\\ x^0 & 2A-6B-6C-3E=13;\end{array}\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\}\begin{array}{l|}A=-1;\\ B=1;\\ C=-2;\\ D=1;\\ E=-3.\end{array}$$

We get that $\cdot \dfrac{P(x)}{Q(x)} = -\dfrac{1}{x-3} + \dfrac{1}{(x+1)^2} - \dfrac{2}{x+1} + \dfrac{x-3}{x^2+2}$.

After this we can integrate separately all obtained rational fractions:

$$\int -\frac{1}{x-3}dx + \int \frac{1}{(x+1)^2}dx - \int \frac{2}{x+1}dx + \int \frac{x-3}{x^2+2}dx =$$

$$-\ln|x-3| - \frac{1}{x+1} - 2\ln|x+1| + \frac{1}{2}\ln|x^2+2| -$$

$$-\frac{3}{\sqrt{2}} arctg \frac{x}{\sqrt{2}} + C .$$

Integration of trigonometric functions

At this point we will have learned more about integrating powers of sine and cosine. If we have the integral in form as

$$\int \sin^m x \cdot \cos^n x dx$$

we should pay attention to the powers of sines (m) and cosines (n), because we will have different ways to solve depending on it.

In this integral if the power on the sines (m) is odd we can strip out one sine, convert the rest to cosines using identity $\cos^2 x + \sin^2 = 1$ and then use the substitution $u = \cos x$. Likewise, if the power on the is odd we can strip out one cosine and convert the rest to sines and use the substitution $u = \sin x$.

Of course, if both powers are odd then we can use either method. However, in these cases it’s usually easier to convert the term with the smaller exponent. The one case we haven’t looked at is what happens if both exponents are even? In this case the technique we used in the first couple of examples simply won’t work and in fact there really isn’t any one set method for doing these integrals.

Each integral is different, and, in some cases, there will be more than one way to do the integral. With that being said most, if not all, of integrals involving products of sines and cosines in which both powers are even can be done using one or more of the following formulas to rewrite the integrand.

$$\frac{1}{2}(1+\cos 2x)=\cos^2 x,\ \frac{1}{2}(1-\cos 2x)=\sin^2 x,\ \cos x\cdot\sin x=\frac{1}{2}\sin 2x.$$

Example 7.7 Find integrals:

a) $\int \sin^3 x\cos^2 xdx$; b) $\int \cos^4 2xdx$.

Solution. a) $\int \sin^3 x\cos^2 xdx=\int \sin^2 x\cos^2 x\sin xdx=$

$$=\left|\begin{array}{c} u=\cos x,\ \ du=-\sin xdx, \\ -du=\sin xdx,\ \ \sin^2 x=1-\cos^2 x=1-u^2 \end{array}\right|=$$

$$-\int u^2du+\int u^4du=\frac{u^5}{5}-\frac{u^3}{3}+C=\frac{\cos^5 x}{5}-\frac{\cos^3 x}{3}+C=-\int\left(1-u^2\right)u^2du=$$

$$-\int u^2du+\int u^4du=\frac{u^5}{5}-\frac{u^3}{3}+C=\frac{\cos^5 x}{5}-\frac{\cos^3 x}{3}+C;$$

b) $\int \cos^4 2xdx=\int \cos^2 2x\cdot\cos^2 2xdx=\int\frac{1}{2}(1+\cos 4x)\cdot\frac{1}{2}(1+\cos 4x)dx=$

$$+\frac{1}{4}\int\cos 4xdx+\frac{1}{4}\int\cos^2 4xdx=\frac{x}{4}+\frac{\sin 4x}{16}+$$

$$\frac{1}{4}\int\left(1+2\cos 4x+\cos^2 4x\right)dx=\frac{1}{4}\int dx+$$

$$+\frac{1}{8}\int(1+\cos 8x)dx=\frac{x}{4}++\frac{\sin 4x}{16}+\frac{x}{8}+\frac{\sin 8x}{64}+C.$$

In general, when we have products of sines and cosines in which both exponents are even we will need to use a series of half angle and/or double angle formulas to reduce the integral into a form that we can integrate. Also, the larger the exponents the more we'll need to use these formulas and hence the messier the problem.

If powers of sine and cosine are negative ($m,n<0$) and their sun is a even number, we should use the substitution as $u=tgx$ $(u=ctgx)$, which allows us to reduce the integrand assignments to the integral of the power function

In the case, where one of the powers of sines or cosines is zero and the other negative, then *a universal trigonometric substitution* $\boxed{u = tg\frac{x}{2}}$ is used. The universal trigonometric substitution could be used also if we have the integral in form as $\int \frac{dx}{a\cos x + b\sin x + d}$ or if we have integral of the rational function of sines and cosines as $R(\sin x, \cos x)$, and the powers of sines and cosines are odd. According to the trigonometric formulas of the half angle, we obtain the following expressions for $\sin x$ and $\cos x$:

$$\sin x = \frac{2u}{1+u^2}; \quad \cos x = \frac{1-u^2}{1+u^2}.$$

Example 7.8 Find integral: $\int \frac{dx}{3+5\cos x}$.

Solution.

$$\int \frac{dx}{3+5\cos x} = \left| \begin{array}{l} u = tg\frac{x}{2}, \ dx = \frac{2du}{1+u^2}, \\ \cos x = \frac{1-u^2}{1+u^2} \end{array} \right| = \int \frac{2du}{(1+u^2)\left(3+5\cdot\frac{1-u^2}{1+u^2}\right)} = \int \frac{2du}{8-2u^2} =$$

$$= \int \frac{2du}{8-2u^2} = \int \frac{du}{4-u^2} = -\int \frac{du}{u^2-4} = -\frac{1}{4}\ln\left|\frac{u-2}{u+2}\right| + C = \frac{1}{4}\ln\left|\frac{2+tg\frac{x}{2}}{2-tg\frac{x}{2}}\right| + C.$$

Lecture 8 DEFINED INTEGRAL AND ITS APPLICATIONS

Integration can be used to find areas, volumes, central points and many useful things. But it is often used to find the area under the graph of a function like this:

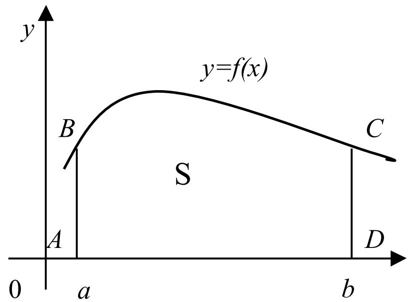


Figure 8.1

$$S = \int_a^b f(x)dx$$

A definite integral is an integral $\int_a^b f(x)dx$ with upper and lower limits. If x is restricted to lie on the real line, the definite integral is known as a Riemann integral (which is the usual definition encountered in elementary textbooks). However, a general definite integral is taken in the complex plane, resulting in the contour integral

$$\int_a^b f(x)dx.$$

So, we have this important thing to remember (Fundamental Theorem of Calculus):

Theorem 8.1 *The fundamental theorem of calculus* establishes the relationship between indefinite and definite integrals and introduces a technique for evaluating definite integrals without using Riemann sums, which is very important because evaluating the limit of Riemann sum can be extremely time-consuming and difficult. The statement of the theorem is: If $f(x)$ is continuous on the interval $[a;b]$, and $F(x)$ is any antiderivative of $f(x)$ on $[a;b]$, then

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a), \; F'(x) = f(x).$$

In other words, the value of the definite integral of a function on $[a;b]$ is the difference of any antiderivative of the function evaluated at the upper limit of integration minus the same antiderivative evaluated at the lower limit of integration. Because the constants of integration are the same for both parts of this difference, they are ignored in the evaluation of the definite integral because they subtract and yield zero. Keeping this in mind, choose the constant of integration to be zero for all definite integral evaluations after

Properties:

1. Adding Functions (or subtraction)

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

2. Reversing the interval

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

3. Interval of zero length

$$\int_a^a f(x)dx = 0$$

4. Adding intervals

$$\int_a^b f(x)dx = \int_a^c f(x)dx \pm \int_c^b f(x)dx\,,\ c \in [a;b].$$

5. A constant factor can be taken as a sign of a definite integral:

$$\int_a^b c \cdot f(x)dx = c \cdot \int_a^b f(x)dx\ ,\ c = const.$$

6. A derivative of indefinite integral is a function

$$\left(\int_a^b f(x) \cdot dx \right)' = f(x)\ .$$

Example 8.1 Calculate the defined integrals:

$$\text{a) } \int_1^2 5^{4-3x}dx\text{; b) } \int_0^2 \left(\sqrt[3]{x} + \frac{7}{x+3} \right) dx\,.$$

Solution:

$$\text{a) } \int_1^2 5^{4-3x}dx = -\frac{1}{3}\cdot\frac{5^{4-3x}}{\ln 5}\Bigg|_1^2 = -\frac{1}{3\ln 5}\left(5^{-2}-5\right) = \frac{24}{75\ln 5};$$

$$\text{b) } \int_0^2\left(\sqrt[3]{x}+\frac{7}{x+3}\right)dx = \int_0^2 x^{1/3}dx + 7\int_0^2\frac{dx}{x+3} = \frac{3x^{4/3}}{4}\Bigg|_0^2 +$$

$$+7\cdot\ln|x+3|\Big\|_0^2 == \frac{3\cdot 2^{4/3}}{4} - 0 + 7\cdot\ln 5 - 7\cdot\ln 3 = \frac{3\cdot\sqrt[3]{2}}{2} + 4\ln\frac{5}{3}.$$

Remark 8.1 Keep in mind that the definite integral is a unique real number and does not represent an infinite number of functions that result from the indefinite integral of a function.

Theorem 8.2 *(The Mean Value Theorem for Definite Integrals)*: If $f(x)$ is continuous on the closed interval $[a;b]$, then at least one number C exists in the open interval $[a;b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

The value of $f(c)$ is called the average or mean value of the function $f(x)$ on the interval $[a;b]$ and

$$f(x) = \frac{1}{b-a}\int_a^b f(x)dx.$$

The numerous techniques that can be used to evaluate indefinite integrals can also be used to evaluate definite integrals. The methods of substitution and change of variables, integration by parts, trigonometric integrals, and trigonometric substitution are illustrated in the following examples.

Example 8.2 Calculate defined integral: $\int_0^1 \frac{x^2dx}{1+x^6}$.

Solution: So, we have function with its derivative, we should use the substitution method, as

$$\int_0^1 \frac{x^2dx}{1+x^6} = \int_0^1 \frac{x^2dx}{1+\left(x^3\right)^2} = \left| \begin{array}{c} t = x^3, \\ dt = 3x^2dx, \\ t_a = 1, \\ t_b = 0 \end{array} \right| =$$

$$\frac{1}{3}\int_0^1 \frac{dt}{1+t^2} = \frac{1}{3} arctgt \Big|_0^1 = \frac{1}{3} arctg\ 1 - \frac{1}{3} arctg\ 0 = \frac{\pi}{12}.$$

Example 8.3 Calculate defined integral: $\int_0^3 \sqrt{9-x^2}dx$.

Solution. So, we have the irrational expression with non-linear radical, we will use the trigonometric substitution

$$\int_0^3 \sqrt{9-x^2}dx = \left| \begin{array}{c} x = 3\sin t,\ t = \arcsin\frac{x}{3}, \\ dx = 3\cos tdt, \\ 9 - x^2 = 9\cos^2 t, \\ t_a = \arcsin\frac{0}{3} = 0, \\ t_b = \arcsin\frac{3}{3} = \frac{\pi}{2} \end{array} \right| =$$

$$\int_0^{\pi/2} \sqrt{9\cos^2 t} \cdot 3\cos tdt = 9\int_0^{\pi/2} \cos^2 tdt = \frac{9}{2}\int_0^{\pi/2}(1+\cos 2t)dt =$$

$$= \frac{9}{2}\int_0^{\pi/2} dt + \frac{9}{2}\int_0^{\pi/2}\cos 2tdt = \frac{9}{2}t\Big|_0^{\pi/2} + \frac{9}{4}\sin 2t\Big|_0^{\pi/2} = \frac{9}{2}\pi.$$

Example 8.4 Calculate defined integral: $\int_0^1 \ln(x+1)dx$.

Solution: We should use integration by part applying this formula to it

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$$

$$\int_0^1 \ln(x+1)dx = \left| \begin{array}{c} u = \ln(x+1), \\ dv = dx, \\ du = \dfrac{dx}{x+1}, \\ v = x \end{array} \right| =$$

$$= x\ln(x+1)\Big|_0^1 - \int_0^1 \frac{xdx}{x+1} = \ln 2 - \int_0^1 \frac{x+1-1}{x+1}dx = \ln 2 -$$

$$- \int_0^1 \frac{x+1}{x+1}dx + \int_0^1 \frac{1}{x+1}dx = \ln 2 - x\Big|_0^1 + \ln(x+1)\Big|_0^1 = 2\ln 2 - 1 .$$

Applications of integration

Area. We have seen how integration can be used to find an area between a curve and the x-axis. With very little change we can find some areas between curves; indeed, the area between a curve and the x-axis may be interpreted as the area between the curve and a second "curve" with equation $y=0$. In the simplest of cases, the idea is quite easy to understand.

$$S = \int_a^b f(x)dx \qquad (8.1)$$

Example 8.5 Find the area below $f_2(x) = x^2 + 4x + 3$ and above $f_1(x) = -x^3 + 7x^2 - 10x + 5$ over the interval $1 \le x \le 2$.

Solution. In figure 8.2 we show the two curves together, with the desired area shaded, then $f_2(x)$ alone with the area under $f_2(x)$ shaded, and then $f_1(x)$ alone with the area under

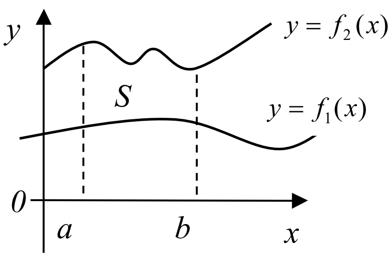


Figure 8.2

$f_1(x)$ shaded. It is clear from the figure that the area we want is the area under $f_2(x)$ minus the area under $f_1(x)$, which is to say

$$S=\int_a^b\left[f_2(x)-f_1(x)\right]dx\,. \qquad (8.2)$$

Note that $a=1$, $b=2$.

It doesn't matter whether we compute the two integrals on the left and then subtract or compute the single integral on the right. In this case, the latter is perhaps a bit easier:

$$S=\int_1^2\left[x^2+4x+3-(-x^3+7x^2-10x+5)\right]dx=$$

$$=\int_1^2\left[x^3-6x^2+14x-2\right]dx=$$

$$=\left(\frac{x^4}{4}-6\frac{x^3}{3}+14\frac{x^2}{2}-2x\right)\Bigg|_1^2=\frac{2^4}{4}-2\frac{2^3}{1}+7\frac{2^2}{1}-2\cdot2-$$

$$-\left(\frac{1^4}{4}-2\cdot1^3+7\cdot1^2-2\right)=\frac{49}{12}(\text{unit}^2)\,.$$

Arc lenght. Therefore, a definite integral is used to calculate the length of an arc of a curve.

Thus, if the function is presented as $y=f(x)$, $x\in[a;b]$, so the arc of the curve will be calculated by formula:

$$l=\int_a^b\sqrt{1+y'^2}\,dx\,; \qquad (8.3)$$

if the function is presented as $y=\phi(t)$, $x=\varphi(t)$ $t\in[\alpha;\beta]$, so the arc of the curve will be calculated by formula:

$$l=\int_\alpha^\beta\sqrt{x'^2+y'^2}\,dt\,; \qquad (8.4)$$

if the function is presented as $\rho = \rho(\varphi)$, $\varphi \in [\varphi_1; \varphi_2]$, so the arc of the curve will be calculated by formula:

$$l = \int_{\varphi_1}^{\varphi_2} \sqrt{\rho^2 + \rho'^2}\, d\varphi. \tag{8.5}$$

Example 8.6 Find the arc length of a single arch of cycloid (Figure 8.3), if it is given in parametric equations:

$$y = 2(1 - \cos t),\ x = 2(t - \sin t).$$

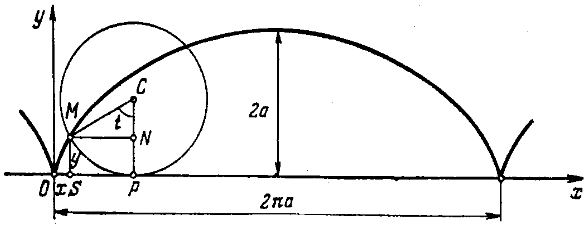


Figure 8.3

Solution: so given function is presented in parametric equations; we will use formula (8.4). First, we will find the derivatives y'_t and x'_t:

$$y'_t = 2\sin t,\ x'_t = 2(1 - \cos t),\ x'^2 + y'^2 = 8 - 8\cos t = 16\sin^2 t/2.$$

A moving point describes one arch of a cycloid when the parameter changes from zero to 2π. Find the length of this arch:

$$l = \int_{\alpha}^{\beta} \sqrt{x'^2 + y'^2}\, dt = \int_{0}^{2\pi} \sqrt{16\sin^2 t/2 dt} = 4\int_{0}^{2\pi} \sin t/2 dt = -8\cos\frac{t}{2}\Big|_0^{2\pi} =$$

$$= -8\cos\frac{2\pi}{2} + 8\cos 0 = 16 \text{ (length units).}$$

Volume. The volume of the body obtained by rotating the figure around the abscissa (figure 8.4) can be calculated by

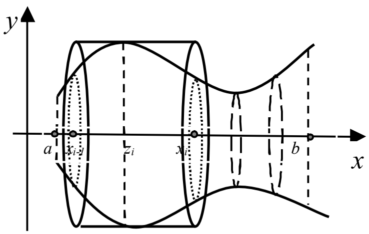


Figure 8.4

$$V_x = \pi \int_a^b f^2(x)dx$$

In the case when a figure is retained round the ordinate axis then the volume should calculate by one of formulas below

$$V_y = \pi \int_c^d f^2(y)dy$$

or

$$V_y = 2\pi \int_a^b x \cdot f(x)dx.$$

Example 8.7 Calculate the volume of the body obtained by rotating a figure bounded by lines

$$y = 4 - x^2, \ y = 3x, \ x = 0.$$

Solution. Construct a figure bounded by lines $y = 4 - x^2$, $y = 3x$, $x = 0$ (figure 8.5). As we can see, the volume of the desired body should be found as the difference between the larger (volume of the outer body) and smaller (volume of the inner body) volumes.

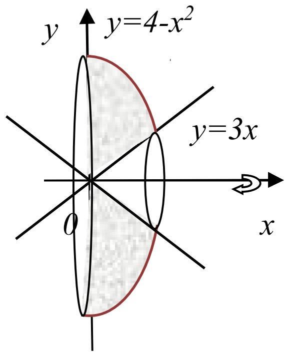


Figure 8.5

But first we find the points of intersection of the graphs of the given functions. To do this, we solve a system of equations

$$\begin{cases} y = 4 - x^2, \\ y = 3x; \end{cases} \begin{cases} x^2 + 3x - 4 = 0, \\ y = 3x; \end{cases} \begin{cases} y_1 = 3, y_2 = -12, \\ x_1 = 1, x_2 = -4. \end{cases}$$

The second point with coordinates $(-4;-12)$ does not satisfy the condition of the problem, because the body is bounded by the *y*-axis. So,

$a = 0$, $b = 1$ are our limits of the integration. We will find it gradually V_{out} and V_{in}, using the formula to the volume calculation:

$$V_{out} = \pi\int_0^1 \left(4 - x^2\right)^2 dx = \pi\int_0^1 \left(16 - 2x^2 + x^4\right)dx = \pi\left(16x - \frac{2x^3}{3} + \frac{x^5}{5}\right)\Bigg|_0^1 =$$

$$= \frac{203\pi}{15} \text{ (units}^3\text{)}; \quad V_{in} = \pi\int_0^1 9x^2 dx = 3\pi x^3\Big|_0^1 = 3\pi \text{ (units}^3\text{)}.$$

So, the desired volume is $V_x = \dfrac{203\pi}{15} - 3\pi = \dfrac{158\pi}{15}$ (units^3).

Therefore, the following formula is finally used to calculate **the surface area of rotation**:

$$S_n = 2\pi\int_a^b f(x)\sqrt{1 + \left(f'(x)\right)^2}\, dx .$$

Example 8.8 Calculate the surface area of a spherical belt formed by rotation around the axis of an arc of a circle with center at the origin and radius 5.

Solution. As is known, the equation of a circle with center at the origin and radius 5 looks like this: $x^2 + y^2 = 25$, then $y^2 = 25 - x^2$, a function that is given implicitly and its derivative is equal to: $yy' = -x$, $y' = \dfrac{-x}{y} = \dfrac{-x}{\sqrt{25 - x^2}}$. Calculate the area of the spherical belt by the formula above:

$$S_n = 2\pi\int_a^b \sqrt{25 - x^2}\sqrt{1 + \left(\frac{-x}{\sqrt{25 - x^2}}\right)^2}\, dx = 2\pi\int_a^b 25dx = 50\pi(b - a).$$

If we consider the expression $(b - a)$ the height H of this belt, then we get, that is $S_n = 50\pi H$, in the case, when $H = 2r = 10$, the surface area of the sphere is equal to $S_n = 500\pi$.

QUESTIONS TO CONSOLIDATE LECTURES

Lecture 1 – 2

1. What is a determinant?

2. What is a minor? 3. What is a cofactor of a determinant element?

4. By what rule the value of the determinant of the n-th order is calculated?

5. Formulate the rules of the “cross” and “triangles” for calculating respectively the determinants of the second and third order

6. What are the basic properties of the determinant? Which of them can we use to calculate it.

8. Will be the value of the determinant changed if the elements of some column are multiplied by 5? If so, how much?

9. Which of the properties can we use to simplify calculation the determinant of any order? Explain your answer.

10. What is a matrix? Which of matrix is called non-degenerate?

11. How are doing the operations of adding (subtracting) matrices and multiplying the matrix by the number?

12. What is the difference between multiplication of the matrix by the scalar and the multiplication of the determinant by the number?

13. How is operation of multiplication of the matrixes carried out? What are the properties of this operation?

14. Which of matrix has determinant? What is an inverse matrix and how is it calculated?

15. Does any matrix have an inverse matrix? Why?

16. How can you check the accuracy of the found inverse matrix?

17. What is the system of linear algebraic equations which has all free terms are zeros? Does such system have a solution? How many?

18. How can we find a solution of a square SLR with an inverse matrix?

19. How to solve the square system of linear equations by Cramer’s rule?

20. How is an arbitrary SLAE solved by the Gaussian elimination method?

21. Is it possible to determine the consistency of the system using the Gaussian elimination method? Explain your answer.

22. How can we know by performing the Gaussian elimination method that the system does not have a solution?

Additional Questions to Self-study Topics

1. What is a general equation of a straight line on a plane?
2. What do know special cases of the general equation of a straight line? How to write the equation of a straight line?
3. How to find the slope of a line? What is happened with line if its slope is zero?
4. How to find the coordinates of the lines intersection point?
5. What is a normal vector?
6. Tell all special case of a plane general equation.
7. What kind of plane equations do you know?
8. What are relationships between two planes in space have?
9. What is the condition of perpendicularity of two planes?
10. What are the differences between a normal vector and a direction vector?
11. What is a circle? What is its standard equation of a circle?
12. What is an ellipse? What are the foci of the ellipse and where are they located?
13. What does the eccentricity of the ellipse characterize?
14. What properties of the ellipse could we learn from its canonical equation?
15. What is a hyperbola? What features do have a hyperbola?
16. What are hyperbola asymptotes?
17. What is a parabola? Give some examples of special case of its graphs.
18. What are polar coordinates? What are the relationships between polar and Cartesian coordinates?

Lecture 3

1. What is the complex number?
2. How can be resented the complex number at the Cartesian frame?
3. What is a conjugate complex number? What is the relationship between a complex and its conjugate number?
4. What can you say about equal complex numbers?
5. What are differences between trigonometric and exponential complex number forms?
6. What can we do operations with the complex numbers?
7. How can we do these operations? Do they have depended on complex numbers forms?

8. What is Euler's basic formula?

9. What is the first and second Muavre's formulas?

10. How can be presented graphically the found complex numbers powers?

11. What is the base algebra theorem?

12. What is a complex variable function? What is the area of its definition? Where does it exist?

13. Explain how to draw the complex variable function? and how to calculate its value?

Lecture 4

1. Explain how do you understand this "limit of the function $f(x)$ from the left and the limit of the function $f(x)$ from the right at the point a"? Is it enough to assert that a function have the limit at the point a?

2. What is an infinitesimal function?

3. Call some properties of infinitesimal and infinitude functions.

4. Which of fundamental limits do know?

5. What kinds of indeterminate forms do you know? Tell us the features of disclosing some of them.

6. Call the types of uncertainties you know and explain how they should be evaluated.

7. What are the consequences of the standard limits do you know? What are infinitesimal small equivalent quantities? How can we use it to evaluate the limits?

Additional Questions to Self-study Topics

1. What is the function continuity? What properties do continuous functions have? Call some elementary continuous functions.

2. What could we tell about the limit of a function at the point at which it has a breaking?

3. What is the function of any variables? How can be it presented graphically?

4. What is the difference between the definition domain of a function of one variable and the definition domain of a function of two (or three) variables?

5. What is the level line? How can a level line be drawn on a surface?

6. What are partial derivatives? Tell us about the peculiarity of finding them.

7. What rules are used for this?
8. What is the geometric meaning of partial derivatives?
9. What is a function gradient? What is its physical meaning?

Lecture 5 – 6

1. What is an increment of a variable?
2. What is a derivative of a function?
3. How to find the derivative of a compose function?
4. In which case should we use the Chain Rule?
5. How to find the derivative of an implicit function?
6. What is a logarithmic differentiation? When could we use it?
7. Explain, how to find the derivative of an implicitly given function.
8. How could be found the derivative of a parametric function?
9. What is the second-order differential? How can we find it?
10. Explain the L'Hospital's rule.
11. Can we use the L'Hospital's rule at all examples or not?
12. How can we use this rule if you need to compute the limit having one of these indeterminate forms $\left[1^{\infty}\right]$, $\left[\infty^{0}\right]$, $\left[0^{0}\right]$?
13. Can you combine the using of the L'Hospital's rule with other ways or previously learned technology?
19. What are conditions of increasing and decreasing the function?
20. What are properties of a critical point?
21. How can be found critical points?
22. What is a necessary condition for the existence of an extremum?
23. What are the exteremal points?
24. How can we find the largest (smallest) value of a function $f(x)$ on a segment $[a,b]$?

Additional Questions to Self-study Topics

1. What is a point of inflexion? How to find it?
2. What does it mean when we say that the curve is concave upwards?
3. What is a sufficient condition for the concavity upwards (downwards) of a function graph?
4. How can we determine that this cure is concave upwards?
5. What is an asymptote? What kind of them do you know?

6. What should we do to find an asymptote to the graph of a function?

7. What is a general scheme of a function analysis?

8. Should we calculate the arbitrary points to draw a graph of a function?

Lecture 7 – 8

1. What is the antiderivative of a function?

2. What is the indefinite integral? Call its properties.

3. What integration methods do you know? Explain in what cases each of them can be applied.

4. Tell us about other integration techniques and what classes of integrals they are used for.

5. What is a definite integral? Call its properties.

6. How will the definite integral change if the upper and lower boundaries of integration are reversed?

7. Could we calculate the approximate value of the defined integral? In which way?

8. What is the value of the definite integral if the boundaries of integration are the same (the contour will be closed)?

9. What is the geometric meaning of the definite integral?

10. What other geometric applications of the definite integral do you know?

Additional Questions to Self-study Topics

1. What is an improper integral?

2. What types of improper integrals do you know? Tell us about the features of calculating each of them.

3. Explain the geometric meaning of the improper integral.

USEFUL RECOURSES

1. Словник з математики (з перекладом російською, українською, англійською, французькою та арабською мовами для іноземних студентів підготовчого відділення) / Харків. нац. ун-т міськ. госп-ва ім. О. М. Бекетова ; уклад. : Г. А. Кузнецова, С. М. Ламтюгова, Ю. В. Ситникова. – Харків : ХНУМГ ім. О. М. Бекетова, 2017. – 56 с.

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13. Mathcenter [Electronic resource] – Available at : http://www.mathcentre.ac.uk/resources/.

14. Mathematic Libretext [Electronic resource]. – Available at : https://math.libretexts.org/Bookshelves/Calculus/

15. Paul's Online Notes [Electronic resource].– Available at : https://tutorial.math.lamar.edu/classes/calcII/calcII.aspx

16. Stephen Roberts. Vector algebra and calculus / [Electronic resource] – Available at : http://www.robots.ox.ac.uk/

17. The Characteristic Polynomial [Electronic resource]. – Available at : http://scipp.ucsc.edu/~haber/ph116A/charpoly_11.pdf

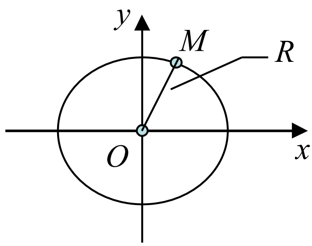
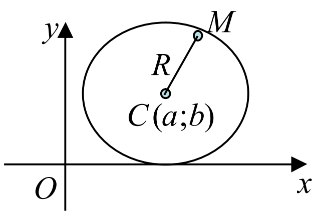
18. Vector algebra [Electronic resource] – Available at : http://www.ncert.nic.in/NCERTS/l/lemh204.pdf

19. WolframMathWorld [Electronic resource]. – Available at : http://mathworld.wolfram.com/.

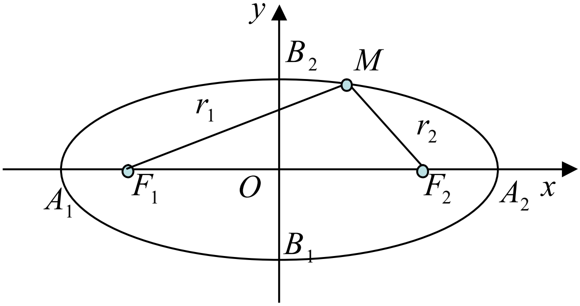
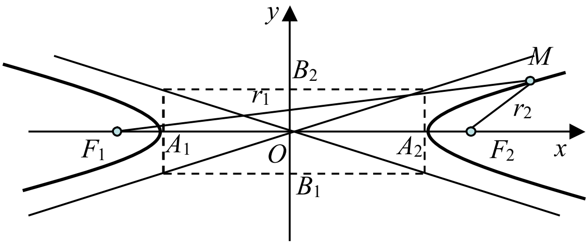
APPENDICES

APPENDIX A

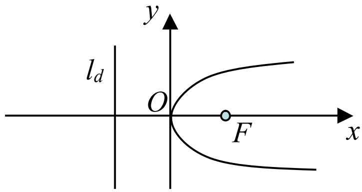
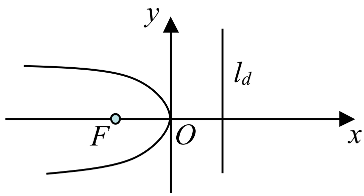
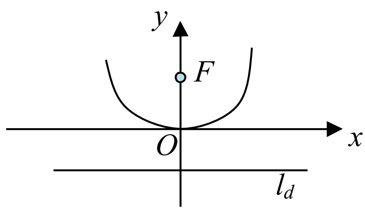
Table A.1 – The second order curve

name	Equation Form	Figure
1	2	3
Circle	$x^2 + y^2 = R^2$	$O(0;0)$ – circle center; MO – radius 
	$(x-a)^2 +$ $+(y-b)^2 = R^2$	$C(a;b)$ – circle center; MC – radius 

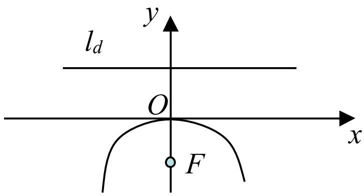
Continued table A.1

1	2	3
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$A_1(-a;0)$, $A_2(a;0)$, $B_1(0;-b)$, $B_2(0;b)$ – ellipse vertices; $F_1(-c;0)$, $F_2(c;0)$ – ellipse focuses 
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$A_1(-a;0)$, $A_2(a;0)$ – real vertices, $B_1(0;-b)$, $B_2(0;b)$ – imagine vertices; $F_1(-c;0)$, $F_2(c;0)$ – hyperbola focuses 

Continued table A.1

1	2	3
Parabola	$y^2 = 2px$	 $l_d : x = -\frac{p}{2}$, $F\left(\frac{p}{2};0\right)$.
	$y^2 = -2px$	 $l_d : x = \frac{p}{2}$, $F\left(-\frac{p}{2};0\right)$
	$x^2 = 2py$	 $l_d : y = -\frac{p}{2}$, $F\left(0;\frac{p}{2}\right)$

Continued table A.1

1	2	3
	$x^2 = -2py$	 $l_d : y = \frac{p}{2}$, $F\left(0; -\frac{p}{2}\right)$

APPENDIX B

Frequently used trigonometric formulas

$$\sin^2 \alpha + \cos^2 \alpha = 1,$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \quad \cot\alpha = \frac{\cos\alpha}{\sin\alpha},$$

$$1 + \tan^2\alpha = \frac{1}{\cos^2\alpha}, \quad 1 + \cot^2\alpha = \frac{1}{\sin^2\alpha},$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha, \quad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha,$$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}, \quad \cos^2\alpha = \frac{1+\cos 2\alpha}{2},$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2},$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2},$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2},$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}.$$

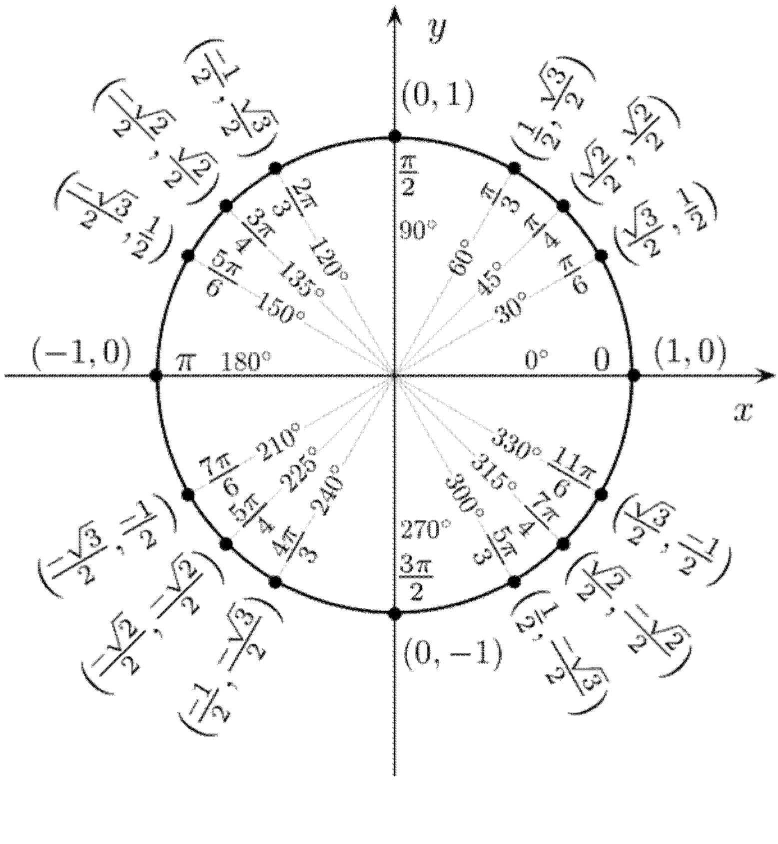
APPENDIX C

Table C.1 – The values of trigonometric functions

Value of angle α		Functions			
degrees	radians	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
0°	0	0	1	0	does not exist ($-\infty$)
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	does not exist ($+\infty$)	0
180°	π	0	-1	0	does not exist ($+\infty$)
270°	$\frac{3\pi}{2}$	-1	0	does not exist ($-\infty$)	0
360°	2π	0	1	0	does not exist ($-\infty$)

PRE-CALCULUS UNIT CIRCLE

In pre-calculus, the unit circle is sort of like unit streets, it’s the very small circle on a graph that encompasses the 0,0 coordinates. It has a radius of 1, hence the unit. The figure here shows all the measurements of the unit circle:



APPENDIX E

Table E.1 – Supplementary integrals

1	$\int \frac{du}{\sin u} = \ln\left\| tg\frac{u}{2} \right\| + C$	2	$\int \frac{du}{\cos u} = \ln\left\| tg\left(\frac{u}{2}+\frac{\pi}{4}\right) \right\| + C$
3	$\int tg\,u\,du = -\ln\left\|\cos u\right\| + C$	4	$\int ctg\,u\,du = \ln\left\|\sin u\right\| + C$
5	$\int sh\,u\,du = ch\,u + C$	6	$\int ch\,u\,du = sh\,u + C$
7	$\int \frac{du}{ch^2 u} = th\,u + C$	8	$\int \frac{du}{sh^2 u} = -cth\,u + C$
9		$\int \sqrt{u^2 \pm a^2}\,du = \frac{1}{2}u\sqrt{u^2 \pm a^2} \pm \frac{1}{2}a^2 \ln\left\| u + \sqrt{u^2 \pm a^2} \right\| + C$	
10		$\int \sqrt{a^2 - u^2}\,du = \frac{1}{a}u\sqrt{a^2 - u^2} + \frac{1}{2}a^2 \arcsin\frac{u}{a} + C$	
11		$\int e^{au}\sin bu\,du = \frac{-be^{au}\cos bu + ae^{au}\sin bu}{a^2+b^2} + C$	
12		$\int e^{au}\cos bu\,du = \frac{ae^{au}\cos bu + be^{au}\sin bu}{a^2+b^2} + C$	

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