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Methodological guidelines
for practical classes,
self-dependent and calculation and graphical works
on the Subject

THEORETICAL MECHANICS
PART 3 DYNAMICS

(for the first year full-time students specialty
192 – Construction and Civil Engineer)

Kharkiv
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2020

Methodological Guidelines for practical classes, self-dependent and calculation and graphical works on the Subject Theoretical Mechanics (Part 3 Dynamics) (for the first year full-time students of the specialty 192 – Construction and Civil Engineer) / O. M. Beketov National University of Urban Economy in Kharkiv ; com. V. P. Shpachuk, A. O. Garbuz, T. A. Suprun, R. L. Turenko. – Kharkiv : O. M. Beketov NUUE, 2020. – 52 p.

Compiler : V. P. Shpachuk,
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Recommended by the department of Theoretical and Structural Mechanics, record № 8 on 19.05.2020.

CONTENTS

1 DYNAMICS of POINT

Dynamics studies the mechanical motion of material objects (material point, system of material points, solids) under the action of the forces applied.

1.1 Axioms of Dynamics (Newton's lows of motion):

First Law (*of inertia*): *A point isolated from other bodies remains at rest or continues to move in straight line with a constant velocity if there is no any force acting on it.*

Second Law (*basic):* *A free point under the acting of any other force is accelerated; the acceleration is in the direction of the force and is directly proportional to the force and inversely proportional to the mass of the point:* $m\overline{a} = \overline{F}$.

Third Law (*law of action and reaction*) *For any action there is always an equal and opposite reaction or, the mutual actions of any two bodies are always equal and oppositely directed.* $\overline{F}_{12} = -\overline{F}_{21}$

Forth Law ***(Principle of superposition)*:** *The resulting acceleration caused by two or more forces is geometrical sum of the accelerations, which would be caused by each force separately:* $\overline{a} = \overline{a_1} + \overline{a_2} + \cdots + \overline{a_n}$.

1.2 Differential forms of free point equation of motion

The *basic equation* of dynamics: $m\overline{a} = \sum_{i=1}^{n} \overline{F}_i$ can be written as a differential equation:

Vector form:

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F} \qquad\qquad m\frac{d\vec{v}}{dt} = \vec{F},$$

where $\boldsymbol{m}$ is mass of the point, $\overline{\boldsymbol{r}}$ is vector-position, which is the function of the time.

Coordinate form (in axis projections):

$$m\ddot{x} = \sum_{i=1}^{n} F_{ix}\,; \quad m\ddot{y} = \sum_{i=1}^{n} F_{iy}\,; \quad m\ddot{z} = \sum_{i=1}^{n} F_{iz}\,,$$

where $\ddot{x} = a_x$; $\ddot{y} = a_y$; $\ddot{z} = a_z$ – projections of accelerations; $\sum_{i=1}^{n} F_{ix}$, $\sum_{i=1}^{n} F_{iy}$, $\sum_{i=1}^{n} F_{iz}$ – algebraic projections of all forces on coordinate axis, acted on point.

Differential equations of a free point motion in projections to *natural coordinate system* τ, n, b (*natural form*):

$$m\frac{dv_\tau}{dt} = \sum_{i=1}^{n} F_{i\tau};\ m\frac{v^2}{\rho} = \sum_{i=1}^{n} F_{in};\ 0 = \sum_{i=1}^{n} F_{ib},$$

where v_τ – velocity projection on tangent; ρ – radius of trajectory curvature at given point; $\sum_{i=1}^{n} F_{i\tau}$, $\sum_{i=1}^{n} F_{in}$, $\sum_{i=1}^{n} F_{ib}$ – algebraic projections of all forces to a *natural* coordinate axis. This equations are used when a point is moving in a circle.

1.3 The two problems of dynamics

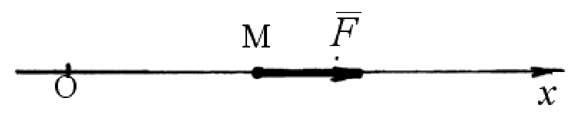
In the first type (*straightforward problem***) –** *the acceleration, motion and mass* are given and *the corresponding forces*, acting on the point must be found. *Solution is reduced to finding the point acceleration.*

In the second type (*inverse* problem**)** – *the applied forces and mass and initial conditions are given and the resulting motion is to be found*. Solution is connected with the integration of differential equations of motion. The general solution of these equations defines the coordinates of a point as a function of time. Constants of integration, when differential equations are integrated, are determined by the initial conditions.

Initial conditions - the value of the coordinates of the point and its first derivatives in time (i. e. projections of the velocity vector to the coordinate axis) at some fixed time point (if $t_0 = 0$): $x|_{t=0} = x_0$, $y|_{t=0} = y_0$, $z|_{t=0} = z_0$, $\dot{x}|_{t=0} = V_{0x}$, $\dot{y}|_{t=0} = V_{0y}$, $\dot{z}|_{t=0} = V_{0z}$.

Problem. The material point with mass **m** moves in a straight line in the direction of the axis under the action of a constant force $\overline{F}$. Find the law of point motion at initial conditions

$$x|_{t=0} = x_0,$$
$$\dot{x}|_{t=0} = V_0.$$



Solution:

1) a differential equation of a material point motion:

$$m\ddot{x} = F \text{, or } \ddot{x} = \frac{F}{m};$$

2) after integrating the differential equation we get:

$$\dot{x} = \frac{F}{m}t + C_1; \qquad x = \frac{F}{m}\frac{t^2}{2} + C_1 t + C_2,$$

where C_1, C_2 – constant of integrations;

3) determine constant of integrations with the initial conditions:

$$V_0 = \frac{F}{m} \cdot 0 + C_1 \qquad \Rightarrow \quad C_1 = V_0;$$

$$x_0 = \frac{F}{m}\frac{0}{2} + C_1 \cdot 0 + C_2 \qquad \Rightarrow \quad C_2 = x_0;$$

4) law of motion we get :

$$x = \frac{F}{m}\frac{t^2}{2} + V_0 t + x_0 .$$

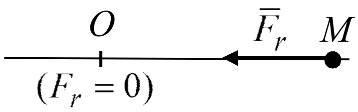
1.4 Vibrations of the material point

Mechanical vibration is the periodic motion of a point displaced from a position of equilibrium. Elastic restoring force, resistance force and external periodic force can act to the material point. The point movement directed at a straight line, which coincides with the action line of the given forces. There are three types of vibration, *free*, *damped* and *forced* oscillations depending on the combination of these forces.

1.4.1 Free Vibration

Restoring force is a force that gives rise to an equilibrium in a physical system. The restoring force magnitude is proportional to the point deviation from the equilibrium position and directed to this position:

Free vibration occurs when the motion is maintained by gravitational or elastic restoring forces.



$$F_r = c \cdot MO,$$

where $\boldsymbol{c}$ is proportional coefficient.

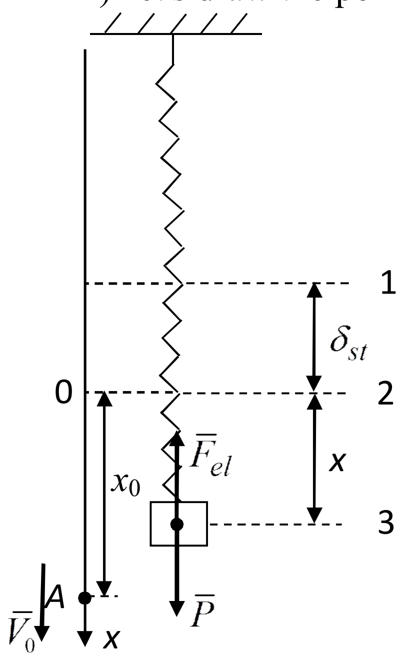
The type of the restoring force is the elastic force or the *spring's stiffness*, through which the elastic properties of the springs are modeled. In linear problems, the force of elasticity corresponds to Hooke's law

$$F_{el} = c \cdot \lambda ,$$

where $\boldsymbol{c}$ - the spring *stiffness*, indicating which force should be applied to the end of the spring for its deformation per unit length; λ - strain of the spring (the difference between the length of the spring in this state (stretched or compressed) and undeformed).

Consider the vertical oscillation of the load m, which is suspended to the spring of stiffness c and carries translation motion. In this case load can be considered a material point of mass m. Define the equation of a point motion:

1) let's draw the point in an arbitrary position on the calculation scheme.



For this, we gradually denote three characteristic levels in the scheme:

1 – free end of non-deformed spring (in this position $F_{el} = 0$);

2 – the static equilibrium position of the point, which motion is studied (this position is lower than level 1). The distance between levels 1 (non-deformed springs) and 2 (static equilibrium points) is called **static strain of the spring** and denoted δ_{st}. In equilibrium position of the point, $\sum_{k=1}^{n} F_{kx} = 0$ or $P - F_{el} = 0$.

The last equality with the deformation $\lambda = \delta_{st}$ of the spring get the form

$$P - c \cdot \delta_{st} = 0.$$

3 – an arbitrary point position during oscillations.

2) in an arbitrary position we shall show the acting forces on the material point, considering it free. It will gravity force $\overline{P}$ of the point and the spring's elasticity $\overline{F}_{el}$;

3) choose a reference system. To start the reference system in the position of the point equilibrium (level 2), and the x-axis should be directed towards the growth of the numbers 1, 2, 3.

4) we write a differential equation of motion of a point.

$$m\overline{a} = \overline{P} + \overline{F}_{el}.$$

On the x-axis $m\ddot{x} = P - F_{el} = P - c(\delta_{el} + x)$

$$m\ddot{x} = P - c \cdot \delta_{st} - cx = -cx$$

$$m\ddot{x} + cx = 0.$$

Differential equation of free vibration:

$$\ddot{x} + \omega_0^2 x = 0,$$

Where $\omega_0^2 = \dfrac{c}{m}$ is *circular frequency*, rad/s.

General solution:

$$x = C_1 \cdot \cos\omega_0 t + C_2 \cdot \sin\omega_0 t$$

where C_1, C_2 – constant of integrations, determined by the initial conditions.

$$\dot{x} = -C_1 \cdot \omega_0 \cdot \sin\omega_0 t + C_2 \cdot \omega_0 \cdot \cos\omega_0 t\,.$$

The *initial conditions*:

$$x\big|_{t=0} = x_0\,, \quad \dot{x}\big|_{t=0} = V_0\,.$$

Substitute

$$\begin{cases} x_0 = C_1 \cdot \cos 0 + C_2 \cdot \sin 0, \\ V_0 = -C_1 \cdot \omega_0 \cdot \sin 0 + C_2 \cdot \omega_0 \cdot \cos 0. \end{cases}$$

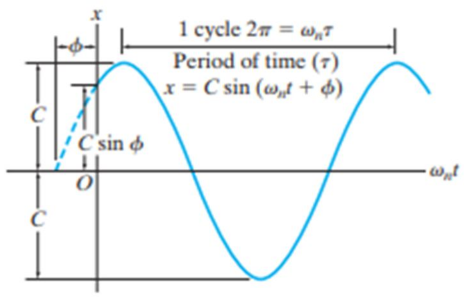
Where constants of the integration are:

$$C_1 = x_0\,, \quad C_2 = \frac{V_0}{\omega_0}\,.$$

Substituting the constants of the integration into expression, we obtain the law of motion:

$$x = x_0 \cdot \cos\omega_0 t + \frac{V_0}{\omega_0} \cdot \sin\omega_0 t\,.$$

This equation is called the equation of free vibration of the material point. It can be presented in the form.



Equation of *free vibration*

$$x = A\sin\left(\omega_0 t + \varphi_0\right)$$

Amplitude of vibration is the maximum displacement of the point from its equilibrium position:

$$A = \sqrt{C_1^2 + C_2^2}\,.$$

Phase angle is $\varphi_0 = arctg\dfrac{C_1}{C_2}$.

From formula (8) parameter A is the **amplitude** of vibration, the argument $\omega_0 t + \varphi_0$ is the **phase** of vibration, and the value φ_0 is the **initial phase**. The parameter ω_0, as the coefficient of time t in the formula of the trigonometric function, is called the **circular frequency** (or **natural frequency**). The *circular frequency* determines the number of oscillations for a time interval of **2π** seconds and is measured in *rad / s* (or *1 / s*).

Period is the time required to complete one cycle:

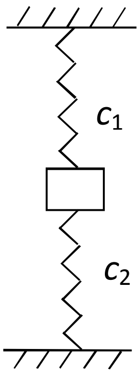
$$T_0 = \frac{2\pi}{\omega_0}. \qquad T = 2\pi\sqrt{\frac{m}{\omega_0}}.$$

The *frequency f* is defined as the number of cycles completed per unit of time, which is reciprocal of the period, and is measured in *hertz* (*Hz*):

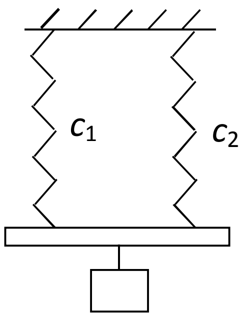
$$f = \frac{1}{T} = \frac{\omega_0}{2\pi}.$$

The mechanical system, considered in the figure, is *conservative*. According to (8), the body (material point) execute a *harmonic vibration motio*n. The circular frequency ω_0 and period T_0 don't depend on the *initial conditions* and are determined by the parameters of the system (*m, c*). The amplitude *A* and the initial phase φ_0 of free oscillations depend on the *initial conditions*.

Springs, which connect the body with the base, can form a system of parallel and series-connected elastic elements. In the case, shown in the figure, the output mechanical schemes need to be reduced into equivalent one elastic element in *free-body diagram*, with equivalent a stiffness c_e.

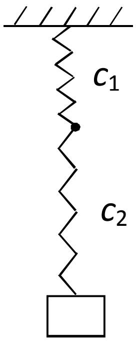


or



An equivalent spring stiffness in the free-body diagram with a parallel connection of the elastic elements will have a stiffness

$$c_e = c_1 + c_2.$$



An equivalent spring stiffness in the free-body diagram with a series-connected elastic elements will have a stiffness

$$c_e = c_1 \cdot c_2 / (c_1 + c_2).$$

When mixed connected three or more elastic elements are successively reduced by two elastic elements will remain until there is one of equivalence stiffness c_e left.

1.4.2 Viscous Damped Vibration

When body is moving in a substance liquid and it's under the acting damping force (viscous friction, such as water, oil, or air). The body moves slowly of through this substance, the resistance force to motion is directly proportional to the body's velocity and has directional to the opposite velocity: $\overline{R} = -b\overline{V}$, where b is called the *coefficient of viscous damping* and has units of *Ns/m*. The viscous properties of elastic bodies are created using résistance forces. Vibrations occurring under the action of the restoring force and the resistance force are ***damped***.

The mechanical system modeling in this case the vibrations of the load (material point) has the same form as in §1.4.1, only the resistance force is added (from the figure we assume that the point moves to the positive direction of the x-axis). Symbols in the figures are the same.

The differential equation of a material point motion will get the form

$$m\overline{a} = \overline{P} + \overline{F}_{el} + \overline{R}$$

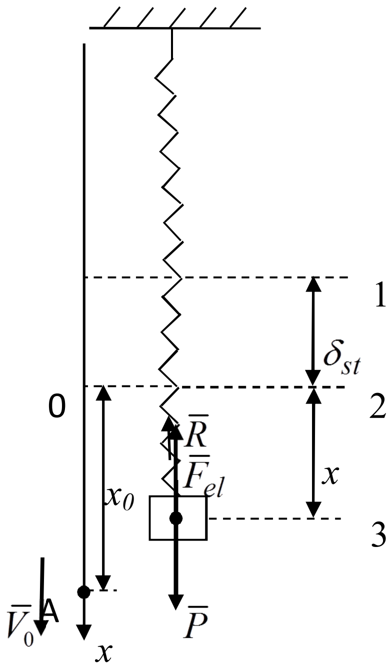
or

$$m\ddot{x} = P - c(\delta_{st} + x) - b\dot{x}$$

or

$$m\ddot{x} + b\dot{x} + cx = 0 .$$

Let's divide all the summands into m and introduce the notation



$$\omega_0^2 = \frac{c}{m},\ 2h = \frac{b}{m},$$

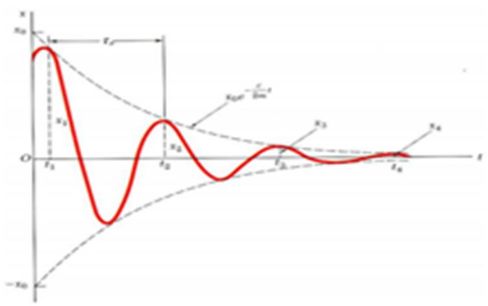
where h – *coefficient of damping vibration*.

So *differential equation damped vibration* gets the form

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x = 0 .$$

The character of the load movement depends on the ratio h and ω_0.

The equation of the load vibration *x(t)* and the first derivative in time $\dot{x}(t)$ in case of *light damped* (*underdamped)* ($h < \omega_0$)



has the form

$$x = e^{-ht}\left(C_1 \cos \omega_0^* t + C_2 \sin \omega_0^* t\right) = A \cdot e^{-ht} \sin\left(\omega_0^* t + \varphi_0\right),$$
$$\dot{x} = -he^{-ht}\left(C_1 \cos \omega_0^* t + C_2 \sin \omega_0^* t\right) + e^{-ht}\omega_0^*\left(-C_1 \sin \omega_0^* t + C_2 \cos \omega_0^* t\right),$$

where C_1, C_2 – constants of integration, which are determined by the initial conditions $x\big|_{t=0} = x_0$, $\dot{x}\big|_{t=0} = V_0$:

$$C_1 = x_0, \quad C_2 = \left(V_0 + hx_0\right)/\,\omega_0^*;$$
$$A = \sqrt{C_1^2 + C_2^2} = \sqrt{x_0^2 + \frac{\left(V_0 + hx_0\right)^2}{\omega_0^{*2}}}; \quad \varphi_0 = arctg\frac{C_1}{C_2} = arctg\frac{x_0 \cdot \omega_0^*}{V_0 + hx_0}; \quad \omega_0^* = \sqrt{\omega_0^2 - h^2}.$$

The mechanical system is called *dissipative* according to the figure, and the relation (13) is called the **equation of damped vibrations**. Therefore, the amplitude of vibrations Ae^{-ht} over time decreases by exponential law to zero. The magnitude ω_0^* is a **damped natural frequency of vibrations** and is connected with a **period of damped vibration** T_0^* by a formula

$$T_0^* = \frac{2\pi}{\omega_0^*}.$$

Heavy damped System. When ($h > \omega_0$), the general solution can be written as

$$x = C_1 e^{p_1 t} + C_2 e^{p_2 t},$$

where $\quad p_1 = -h + \sqrt{h^2 - \omega_0^2}; \qquad p_2 = -h - \sqrt{h^2 - \omega_0^2};$

This equation describes the *aperiodic damped motion*. The coordinate of the load is reduced in $t \to 0$ monotonously, and the load approaches the static equilibrium position.

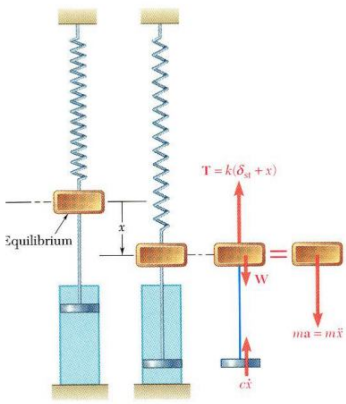
Critical damping: $h = \omega_0$. The law of the load movement will be as

$$x = e^{-ht}\left(C_1 + C_2 \cdot t\right)$$

The considered motion is also *aperiodic damped* ($x \to 0$ if $t \to \infty$). Motion corresponding to this solution is *no vibrating*.

1.4.3 Damped Free Vibrations

All vibrations are *damped* to some degree by forces due to *dry friction*, *fluid friction*, or *internal friction*.



With *viscous damping* due to fluid friction,

$$ma = \sum F$$

$$m\ddot{x} = P - c(\delta_{st} + x) - b\dot{x}$$
$$m\ddot{x} + b\dot{x} + cx = 0$$

Differential equation *Damped Free Vibrations*

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x = 0,$$

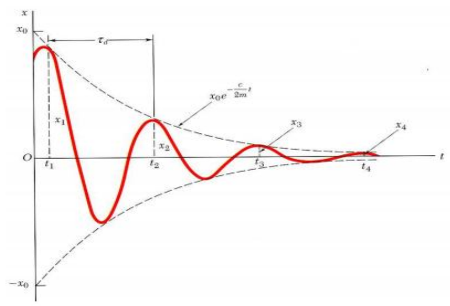
where $\omega_0^2 = \dfrac{c}{m}$, $2h = \dfrac{b}{m}$,

h – damped coefficient.

Solutions:

• *light damping:* $(h < \omega_0)$

$$x = e^{-ht}\left(C_1 \cos\omega_0^* t + C_2 \sin\omega_0^* t\right) = A \cdot e^{-ht} \sin\left(\omega_0^* t + \varphi_0\right)$$



$$C_1 = x_0,\quad C_2 = \left(V_0 + hx_0\right)/\omega_0^*;$$

$$A = \sqrt{C_1^2 + C_2^2} = \sqrt{x_0^2 + \frac{\left(V_0 + hx_0\right)^2}{\omega_0^{*2}}}$$

$$\varphi_0 = arctg\frac{C_1}{C_2} = arctg\frac{x_0 \cdot \omega_0^*}{V_0 + hx_0}$$

$\omega_0^* = \sqrt{\omega_0^2 - h^2}$ - *damped frequency*.

- *Critical damping*: $h = \omega_0$

$$x = e^{-ht}\left(C_1 + C_2 \cdot t\right).$$

($x \to 0$ при $t \to \infty$) double roots, no vibratory motion.

- *Heavy damping*: ($h > \omega_0$)

$$x = C_1 e^{p_1 t} + C_2 e^{p_2 t},$$

where $p_1 = -h + \sqrt{h^2 - \omega_0^2}$; $p_2 = -h - \sqrt{h^2 - \omega_0^2}$; negative roots, no vibratory motion.

1.4.4 Undamped Forced Vibrations

Vibrations which occur under the action of *restoring force* and *periodic force*, which varies over time, are called **forced vibration**. These vibrations arise in the case of direct action on the point of *exciting force* (force excitation of oscillations), when moving the point according to the periodic support displacement of a system (kinematic excitation oscillations), as well as the mutual displacement of masses, which constitute a mechanical system connected with spring base.

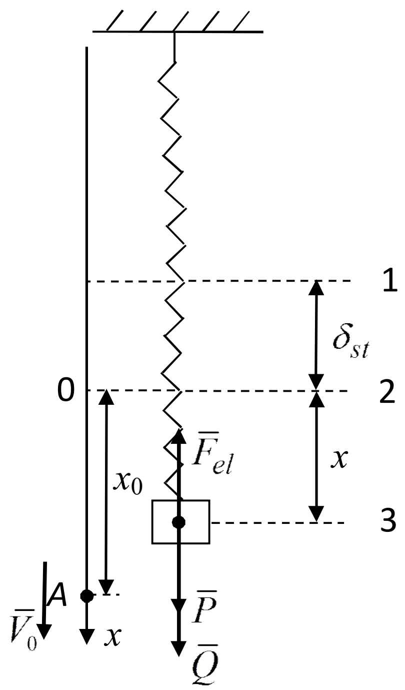
Forced Vibrations

A *force* $Q(t)$ which changes in time according to a given law is called *exciting force*. Let's consider a simpler important case where force varies according to a harmonious law

$$Q = H \cdot \sin \omega_f t,$$

where H – amplitude of exciting force; ω_f – the forcing frequency.

The mechanical system modeling in this case the vibrations of the load (material point) has the same form as in §1.4.1, only the exciting force is added (in the figure the exciting force directed to the positive direction of the x-axis). Symbols in the figures are the same.



The differential equation of a material point motion will get the form

$$m\overline{a} = \overline{P} + \overline{F}_{el} + \overline{Q}$$

or $\quad m\ddot{x} = mg - c(\delta_{st} + x) + H \sin \omega_f t,$

or $\quad m\ddot{x} + cx = H \sin \omega_f t.$

Let's divide all the summands into m and introduce the notation

$$\omega_0^2 = \frac{c}{m},\ h_f = \frac{H}{m}.$$

So *differential equation of forced vibration* gets the form

$$\ddot{x} + \omega_0^2 x = h_f \sin \omega_f t.$$

This equation (19) is a linear nonhomogeneous second-order differential equation. The general solution consists of $x=x_1+x_2$, where x_1 is a complementary solution, and x_2 is a particular solution.

The general solution is therefore the sum of two sine functions having different frequencies. In determining the partial solution let's consider three cases:

$$\omega_f \neq \omega_0; \qquad \omega_f \approx \omega_0; \qquad \omega_f = \omega_0:$$

1) $\omega_f \neq \omega_0$ (frequency of exciting force and circular frequency are different). In this case, the partial solution has the form

$$x_2 = \frac{h_f}{\omega_0^2 - \omega_f^2} \sin \omega_f t$$

and the equation of vibration of the load *x(t)* and the first derivative in time $\dot{x}(t)$ have the form

$$x = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{h_f}{\omega_0^2 - \omega_f^2} \sin \omega_f t \ ,$$

$$\dot{x} = -C_1 \cdot \omega_0 \cdot \sin \omega_0 t + C_2 \cdot \omega_0 \cdot \cos \omega_0 t + \frac{h_f \cdot \omega_f}{\omega_0^2 - \omega_f^2} \cdot \cos \omega_f t .$$

where C_1, C_2 – constants of integration, which are determined by the initial conditions and in the case $x\big|_{t=0} = x_0$, $\dot{x}\big|_{t=0} = V_0$ are

$$C_1 = x_0, \ C_2 = \frac{V_0}{\omega_0} - \frac{h_f \cdot \omega_f}{\omega_0 \left(\omega_0^2 - \omega_f^2 \right)} .$$

Equation (20) is the **equation of forced vibrations**. The law of motions of the load with forced vibrations is *biharmonic* (two-frequency ones) – there is an overlay of free (own) oscillations of the conservative mechanical system (the first two additions in formula (20)) to the forced vibrations with a frequency of pereodic force (the last addition in formula (20)). The feature of such vibrations is that their oscillations are also excited under zero initial conditions. In real systems, their own vibrations due to the resistance of the movement quickly fade out and there are purely forced oscillations. The circular frequency of purely forced vibrations ω_f from system parameters and initial conditions doesn't depend, but it's determined by the parameters of exciting force. **Amplitude of purely forced vibrations** $A_f = \left| \frac{h_f}{\omega_0^2 - \omega_f^2} \right|$ also doesn't depend on the initial conditions but it's determined by the parameters of the exciting force (H, ω_f) and mechanical system (m, c).

2) $\omega_f \approx \omega_0$ (frequency of exciting force and own frequency are close).

Supposing the initial conditions are zero ($x_0 = 0, V_0 = 0$). Then, considering $\omega_f / \omega_0 \approx 1$, but $\omega_0^2 - \omega_f^2 \neq 0$ i $\omega_0 + \omega_f \approx 2\omega_f$, from equation (20) we get

$$x \approx \frac{h_f}{\omega_0^2 - \omega_f^2} \cdot (\sin\omega_f t - \sin\omega_0 t) \approx 2 \cdot \frac{h_f}{\omega_0^2 - \omega_f^2} \cdot \sin(\frac{\omega_f - \omega_0}{2} \cdot t) \cdot \cos\omega_f t\,.$$

This movement is called a **beating**: oscillatory motion that occurs with frequency ω_f exciting force and amplitude which is periodic (with frequency $(\omega_f - \omega_0)/2$) time function.

3) $\omega_f = \omega_0$ (frequency of exciting force and own frequency are coincide). In this case, the partial solution has the form

$$x_2 = -\frac{h_f t}{2\omega_f} \cos\omega_f t$$

and the equation of vibration of the load *x(t)*

$$x = C_1 \cdot \cos\omega_0 t + C_2 \cdot \sin\omega_0 t - \frac{h_f}{2\omega_f} t \cdot \cos\omega_f t\,.$$

It's easy to notice the amplitude of forced oscillations $\frac{h_f \cdot t}{2\omega_f}$ increases directly proportionally to time and with $t \to \infty$ unlimited increases. The phenomenon of unlimited growth of the amplitude of forced oscillations is **resonance**, and equality $\omega_B = \omega_0$ is a **condition of resonance**. In practical constructions, the phenomenon of resonance takes place as the reason for their destruction.

Periodic Support Displacement

Periodic Support oscillation movement takes place when the base at the point of attachment starts oscillation movement due to the law late

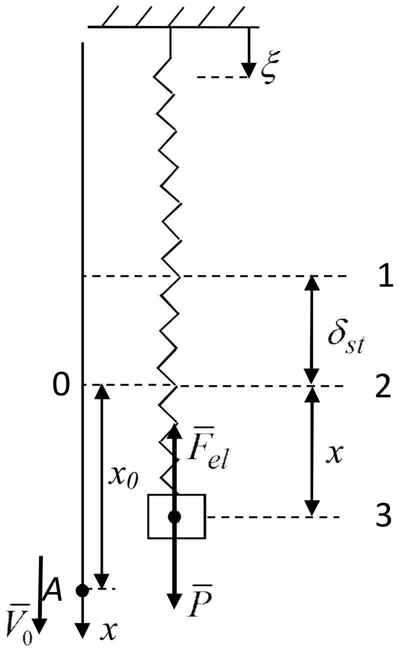
$$\xi = a_\xi \cdot sin\omega_f t,$$

where a_ξ – the amplitude of base movement, $\boldsymbol{\omega_f}$ – its circular frequency.

The mechanical system modeling in this case the vibrations of the load (material point) has the same form as in §1.4.1, only *periodic support displacement* ξ is added (considering in the figure the displacement directed to the positive direction of the *x*-axis). Symbols in the figures are the same.

The differential equation of a material point motion will get the form

$$m\bar{a} = \bar{P} + \bar{F}_{el}$$



or $\qquad m\ddot{x} = mg - c(\delta_{st} + x - \xi)$,

or $\qquad m\ddot{x} + cx = c \cdot a_{\xi} \cdot \sin\omega_f t$.

To divide all the summand into m and introduce the notation

$$\omega_0^2 = \frac{c}{m},\ h_f = \frac{c \cdot a_{\xi}}{m}.$$

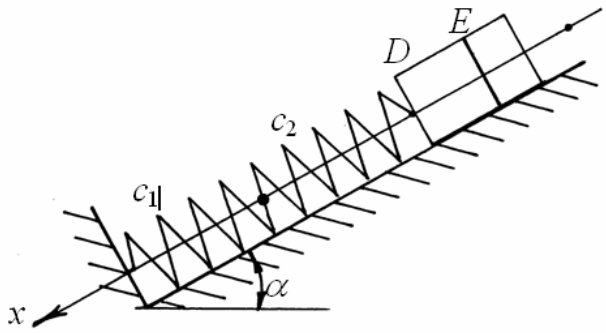
So *differential equation of excited vibration* gets the form

$$\ddot{x} + \omega_0^2 x = h_f \sin\omega_f t.$$

Consequently, the properties of forced vibrations in the kinematic excitation will be the same as in force.

1.4.5 Examples of solving research problems vibration motion of a material point

Problem 1. Determine the equation of the vibration motion of the load D along a smooth inclined plane in the direction of the x-axis, which coincides with the axis of two series of connected springs.



Given. The system of D ($m_D = 20$ kg) and E ($m_E = 10$ kg) put on springs is in the position of static equilibrium. In time $t = 0$, the load E is take off from the load D. At the same time, the load D adds an initial velocity $V_0 = \dot{x}_0 = 0{,}1\ m/s$ at positive direction reference x coordinates. The stiffness of the springs are equal $c_1 = c_2 = 400\ N/cm$. Angle $\alpha = 30^0$.

Solution. Convert the given mechanical system to the free-body diagram with one spring element (see the methodology of § 1.4.1) and represent it in the figure, where 1,2,3 are determined by the free end of the undeformed spring respectively, the position of the static balance of the load D and its arbitrary position during the vibrations, point A – the position of the load D at the moment of oscillation, δ_{st_D} static deformation of the spring under the action of the load D, D, $\overline{P}_D, \overline{N}, \overline{F}_{el}$ the weight of the load, the normal surface reaction and the spring strength of the spring, accordingly.

In this scheme, in the position of the static equilibrium of the load D, the spring strength $\overline{F}_{el}$ does not balance the weight of the load $\overline{P}_D$, but only its component $P_{Dx} = P_D \cdot \sin\alpha$ in the direction of the axis *Ox*, which coincides with the axle of the spring.

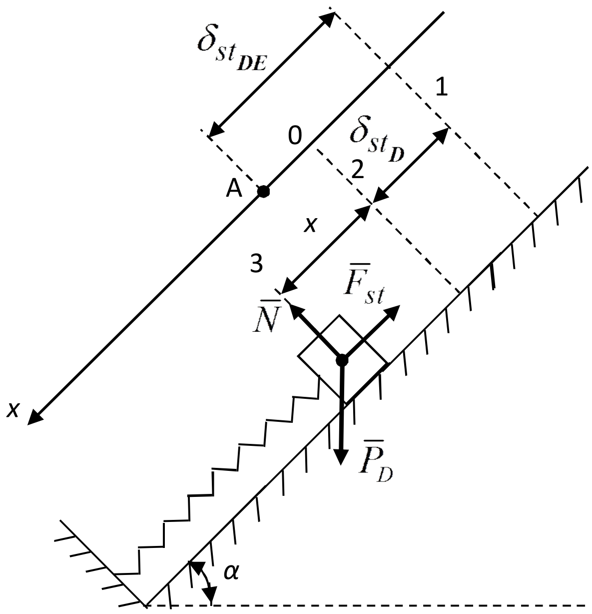
We find an equivalent stiffness *c* (show § 1.4.1):

$$c = \frac{c_1 \cdot c_2}{c_1 + c_2} = \frac{400 \cdot 400}{400 + 400} = 200 \text{ N/cm} = 2 \cdot 10^4 \text{ N/m}.$$

We write a differential equation of the point's motion:

$$m_D \ddot{x} = P_{Dx} - F_{el} = P_D \sin\alpha - c(\delta_{st_D} + x) = P_D \sin\alpha - c\delta_{st_D} - cx = -cx,$$

and convert it into a standard form



$$m_D \ddot{x} + cx = 0$$

or

$$\ddot{x} + \frac{c}{m_D} x = 0,$$

denoted $\omega_0^2 = \dfrac{c}{m_D}$,

so $\ddot{x} + \omega_0^2 x = 0$.

The received equation has the form of the differential equation of free vibrations. It solution – function $x(t)$, and it first derivative in time $\dot{x}(t)$ has form

$$x = C_1 \cdot \cos\omega_0 t + C_2 \cdot \sin\omega_0 t,$$

$$\dot{x} = -C_1 \cdot \omega_0 \cdot \sin\omega_0 t + C_2 \cdot \omega_0 \cdot \cos\omega_0 t.$$

To determine the constant of integration we take the initial conditions (the coordinate of point A (see free-body diagram) and the projection of the initial velocity $V_0 = V_A = 0{,}1$ m/s):

$$x\big|_{t=0} = \delta_{st_{DE}} - \delta_{st_D} = \delta_{st_E}, \quad \dot{x}\big|_{t=0} = V_0$$

Substitute them in expression $x(t)$ i $\dot{x}(t)$

$$\begin{cases} \delta_{cm_E} = C_1 \cdot \cos 0 + C_2 \cdot \sin 0, \\ V_0 = -C_1 \cdot \omega_0 \cdot \sin 0 + C_2 \cdot \omega_0 \cdot \cos 0. \end{cases}$$

So the constant of integration will be

$$C_1 = \delta_{st_E}, \quad C_2 = \frac{V_0}{\omega_0}.$$

Substituting them into function $x(t)$, we obtain the law of weight motion in a general form

$$x = \delta_{st_E} \cdot \cos\omega_0 t + \frac{V_0}{\omega_0} \cdot \sin\omega_0 t.$$

Calculate the values of constant parameters:
the value of static strain

$$\delta_{st_E} = \frac{m_E g \cdot \sin\alpha}{c} \approx \frac{10 \cdot 10 \cdot 0,5}{2 \cdot 10^4} = 0,25 \cdot 10^{-2} \text{ m},$$

the value own frequency $\omega_0 = \sqrt{\frac{c}{m_D}} = \sqrt{\frac{2 \cdot 10^4}{20}} = \sqrt{1000} \approx 31,6 \ \frac{\text{rad}}{\text{s}},$

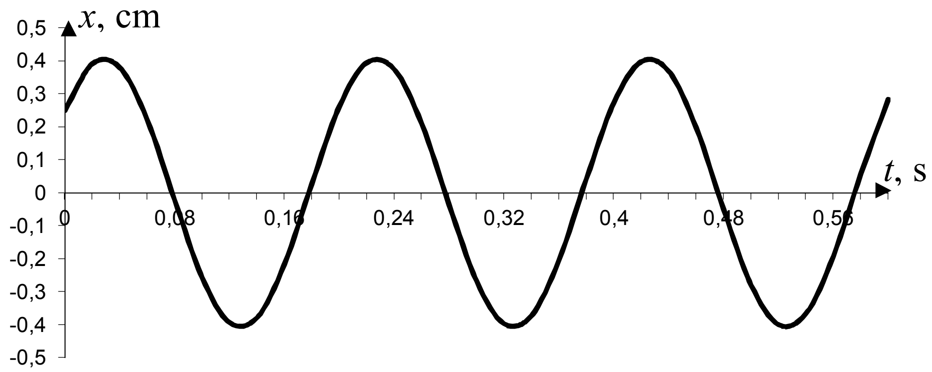
$$\frac{V_0}{\omega_0} = \frac{0,1}{31,6} \approx 0,32 \cdot 10^{-2} \ m.$$

Law of motion:

$$x = 0,25 \cdot 10^{-2} \cdot \cos 31,6\text{t} + 0,32 \cdot 10^{-2} \cdot \sin 31,6\text{t}\,(m).$$

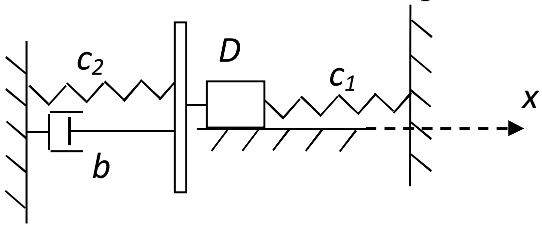
Checking: If $t = 0$ we get $x\big|_{t=0} = 0,25 \cdot 10^{-2}$ m, which coincides with the value of the earlier defined initial condition x_0.

Answ: Weight performs free vibration according to the law $x = 0,25 \cdot 10^{-2} \cdot \cos 31,6\text{t} + 0,32 \cdot 10^{-2} \cdot \sin 31,6\text{t}$ (m), with a circular frequency $\omega_0 = 31,6 \ \frac{\text{rad}}{\text{s}}$, period $T_0 = 0,2$ s and amplitude $A = \sqrt{C_1^2 + C_2^2} = 0,41 \cdot 10^{-2}$ m. Diagram ratio coordinate x in time t



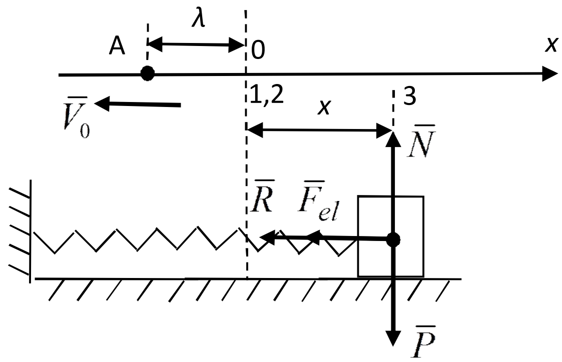
Problem 2. Find the equation of the vibration motion of the load D in the direction of the horizontal axis x, considering it as material point. The bar that connects the spring and damper is weightless. The bar is vertical position in a state of rest, motion considered as translational.

Condition of the Problem. Mass of load m_D = 4 kg. Stiffness of two springs c_1 = 2 N/cm, c_2 = 3 N/cm, damped coefficient $b = 16$ Ns/m.



Before moving, the load is rejected on the length $\lambda = 3$ cm in the direction of the negative reference coordinates x in time $t = 0$, giving the load an initial velocity $V_0 = 0{,}6$ m/s, directed to left.

Solution. Convert the original mechanical system into a free-body diagram with one elastic element (see § 1.4.1) and give it into figure, where levels 1,2,3 show the free end of the non-deformed spring, the position of the static equilibrium of the load D (which for a horizontal spring coincides with level 1; in the case $\delta_{st} = 0$) and its arbitrary position during vibrations respectively. Point A – position of load D at the moment of start vibration, $\overline{V}_0$ – initial velocity vector, $\overline{P}, \overline{N}, \overline{F}_{el}, \overline{R}$ – the force of gravity of the load, the normal reaction of the surface, the elastic force of the spring and the resistance force in respectively



Determine equivalent stiffness c:

$c = c_1 + c_2 = 2 + 3 = 5$ N/cm =

= $5 \cdot 10^2$ N/m.

Wright a differential equation of the point's motion:

$m_D \ddot{x} = -F_{el} - R = -cx - b\dot{x},$

and convert it into a standard form:

$$\ddot{x} + \frac{b}{m_D}\dot{x} + \frac{c}{m_D}x = 0.$$

Noted $\omega_0^2 = \dfrac{c}{m_D}$, $2h = \dfrac{b}{m}$, so $\ddot{x} + 2h\dot{x} + \omega_0^2 x = 0$.

The resulting equation has the form of the differential equation of free vibrations considering of damped vibration. We calculate the values of *circular frequency* of free vibrations:
$$\omega_0 = \sqrt{\frac{c}{m_D}} = \sqrt{\frac{5 \cdot 10^2}{4}} = \sqrt{125} \approx 11{,}2\ \frac{\text{rad}}{\text{s}},$$

and damped coefficient:
$$h = \frac{b}{2m} = \frac{16}{2 \cdot 4} = 2\ \frac{\text{rad}}{\text{s}}.$$

Comparing these parameters, we have $h < \omega_0$ a case of light damped vibration. In this case, the function $x(t)$ and its first derivative in time $\dot{x}(t)$ have the form

$$x = e^{-ht} \cdot \left(C_1 \cos\omega_0^* t + C_2 \sin\omega_0^* t\right),$$

$$\dot{x} = -h \cdot e^{-ht} \cdot (C_1 \cdot \cos\omega_0^* t + C_2 \cdot \sin\omega_0^* t) + e^{-ht} \cdot \omega_0^* \cdot (-C_1 \cdot \sin\omega_0^* t + C_2 \cdot \cos\omega_0^* t).$$

To determine the constants of integration we take the initial conditions (the coordinate of point A (see free-body diagram) and the projection of the initial velocity):

$$x\big|_{t=0} = -\lambda, \quad \dot{x}\big|_{t=0} = -V_0$$

and substitute them for function $x(t)$ i $\dot{x}(t)$

$$\begin{cases} -\lambda = e^0 \cdot (C_1 \cdot \cos 0 + C_2 \cdot \sin 0), \\ -V_0 = -h \cdot e^0 \cdot (C_1 \cdot \cos 0 + C_2 \cdot \sin 0) + e^0 \cdot \omega_0^* \cdot (-C_1 \cdot \sin 0 + C_2 \cdot \cos 0). \end{cases}$$

So constants of integration

$$C_1 = -\lambda, \; C_2 = \frac{-V_0 - h\lambda}{\omega_0^*}.$$

in a general form

$$x = -e^{-ht} \cdot \left(\lambda \cdot \cos\omega_0^* t + \frac{V_0 + h\lambda}{\omega_0^*} \cdot \sin\omega_0^* t\right).$$

$$\omega_0^* = \sqrt{\omega_0^2 - h^2} = \sqrt{125 - 4} = \sqrt{121} = 11 \; \frac{\text{rad}}{\text{s}},$$

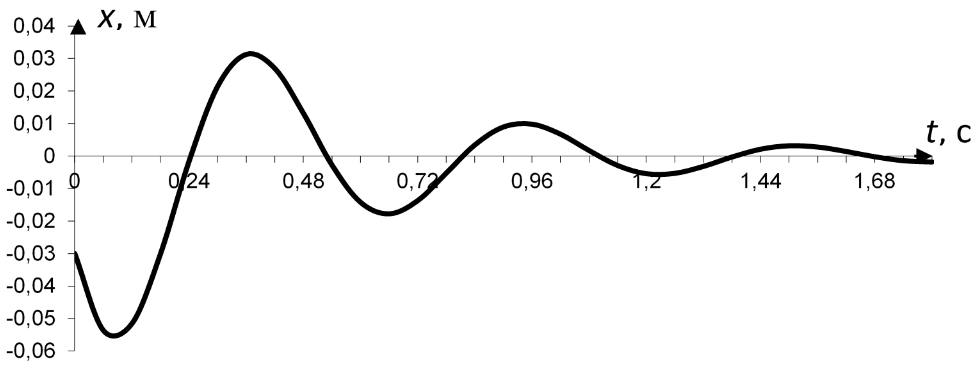
$$\frac{V_0 + h\lambda}{\omega_0^*} = \frac{0,6 + 2 \cdot 0,03}{11} = \frac{0,66}{11} \approx 0,06 \text{ m.}$$

So equation of motion is:

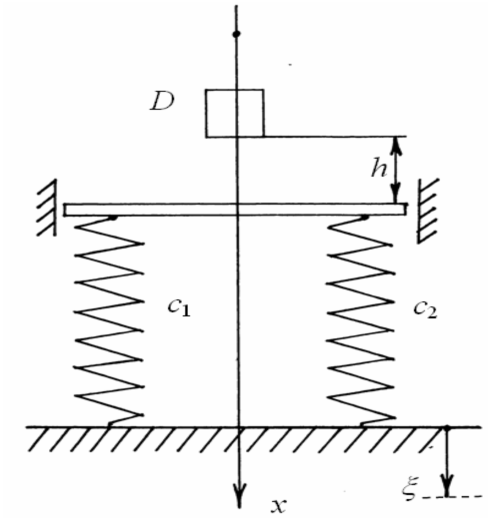
$$x = -\mathrm{e}^{-2t} \cdot (0,03 \cdot \cos 11t + 0,06 \cdot \sin 11t)\,(m).$$

Checking: at time $t = 0$ get $x\big|_{t=0} = -0,03$ m, which coincides with the value of the initial condition x_0.

Answ: the load performs *damped vibration* according to the law $x = -\mathrm{e}^{-2t} \cdot (0,03 \cdot \cos 11t + 0,06 \cdot \sin 11t)$ (m), with *damped natural frequency* ω_0^* = 11 rad/s and period T_0^* = 0,58 s. The diagram ratio the coordinate x of time t is

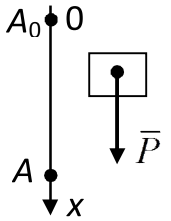


Problem 3. Find the equation of the vibration motion the load D in the direction of the x-axis from the time it touches the plate, considered, the load from the plate isn't separated with further motion. The plate, which is horizontal position in a state of rest, is weightless. Plate and bases motion are considered translationally.



Condition of the Problem. Come down without initial velocity distance $h = 0{,}2$ m, load D ($m_D = 20$ kg) connecting with plate at time $t = 0$, which connects the system of two non-deformed parallel-connected springs. Springs have coefficients of stiffness and viscous damping c_1 = 100 N/cm, c_2 = 200 N/cm, $b_1 = b_2 = 0$. At the same time, the basis begins to move by law $\xi = a_\xi \cdot \sin\omega_f t = 0{,}5 \cdot \sin 30t$ (cm).

Solution: First, consider the auxiliary problem of the free fall of the material point (load) from the height $h = A_0 A$ to the time of contact with the plate.



Free-body diagram this problem shown in the figure. Differential equation motion is: $m_D\ddot{x} = P$, or $m_D\ddot{x} = m_D g$, or $\ddot{x} = g$. By integrating both parts of the last equation in time, we get

$$\begin{cases} \dot{x} = gt + C_1, \\ x = gt^2/2 + C_1 t + C_2. \end{cases}$$

If initial conditions are zero, so the constants of integration are zero. Writing functions $\dot{x}, x$ at the final point A the segment A_0A, we have a system algebraic equations

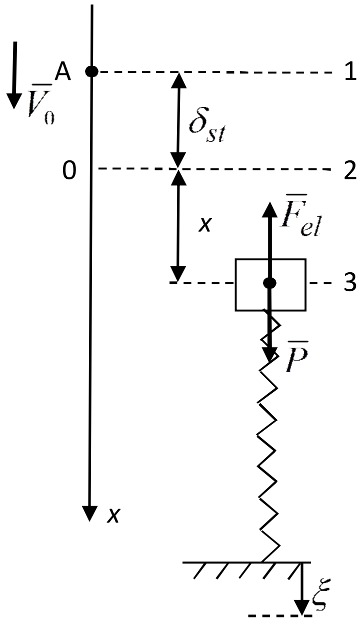
$$\begin{cases} V_A = g\tau, \\ h = g\tau^2/2, \end{cases}$$

where, excluding the time of motion on the segment of the fall τ, we get the formula for the point velocity at the time of contact with the plate $V_A = \sqrt{2gh} \approx \sqrt{2 \cdot 10 \cdot 0{,}2} = 2$ (m/s).

Consider the basic problem of the material point variation (load).
Convert the original mechanical system into a free-body diagram with one elastic element (show § 1.4.1) and give it into figure, where levels 1, 2, 3 determine the free end of the non-deformed spring, the position of the static equilibrium of the load D

(which for a horizontal spring coincides with level 1; in the case $\delta_{st} = 0$) and its arbitrary position during vibrations respectively. Point A is the position of the load at the time of touching the plate, $\delta_{ст}$ is static strain of the spring under the action of the load, $\overline{P}, \overline{F}_{el}$ - the force of gravity the load and the elastic force of the spring respectively, ξ is the direction of kinematic excitation at the point of fixing the spring to the moving support. Determine equivalent stiffness c:

$$c = c_1 + c_2 = 100 + 200 = 300 \text{ N/cm} = 3\cdot 10^4 \text{ N/m}.$$



Write differential equation point motion:

$$m_D\ddot{x} = P - F_{el} = P - c(\delta_{st} + x - \xi) =$$
$$= P - c\delta_{st} - cx + c\xi = -cx + ca_\xi \sin\omega_f t,$$

and convert it to standard form:

$$m_D\ddot{x} + cx = ca_\xi \sin\omega_f t$$

or
$$\ddot{x} + \frac{c}{m_D}x = \frac{ca_\xi}{m_D}\sin\omega_f t,$$

noted $\omega_0^2 = \dfrac{c}{m_D}$, $h_f = \dfrac{c\cdot a_\xi}{m_D}$,

$$\ddot{x} + \omega_0^2 x = h_f \sin\omega_f t.$$

The got equation has the form differential equation of kinematic oscillations. Its solution – function $x(t)$, and its derivative in time $\dot{x}(t)$ are

$$x = C_1\cdot\cos\omega_0 t + C_2\cdot\sin\omega_0 t + \frac{h_f}{\omega_0^2 - \omega_f^2}\cdot\sin\omega_f t,$$

$$\dot{x} = -C_1\cdot\omega_0\cdot\sin\omega_0 t + C_2\cdot\omega_0\cdot\cos\omega_0 t + \frac{h_f\cdot\omega_f}{\omega_0^2 - \omega_f^2}\cdot\cos\omega_f t.$$

For determination constants of integration C_1, C_2 write initial conditions (coordinate of point A (show free body diagram) and projection its initial velocity $V_0 = V_A = 2$ m/s):

$$x\big|_{t=0} = -\delta_{st}, \quad \dot{x}\big|_{t=0} = V_0$$

and substitute them into the equations $x(t)$ i $\dot{x}(t)$

$$\begin{cases} -\delta_{st} = C_1 \cdot \cos 0 + C_2 \cdot \sin 0 + \dfrac{h_f}{\omega_0^2 - \omega_f^2} \cdot \sin 0, \\ V_0 = -C_1 \cdot \omega_0 \cdot \sin 0 + C_2 \cdot \omega_0 \cdot \cos 0 + \dfrac{h_B \cdot \omega_B}{\omega_0^2 - \omega_B^2} \cdot \cos 0. \end{cases}$$

So constants of integration C_1, C_2 are

$$C_1 = -\delta_{st}, \quad C_2 = \frac{1}{\omega_0} \cdot (V_0 - \frac{h_f \cdot \omega_f}{\omega_0^2 - \omega_f^2}).$$

Substituting them in a function $x(t)$, we get the equations of motion load in general form

$$x = -\delta_{st} \cdot \cos \omega_0 t + \frac{1}{\omega_0} \cdot (V_0 - \frac{h_f \cdot \omega_f}{\omega_0^2 - \omega_f^2}) \cdot \sin \omega_0 t + \frac{h_f}{\omega_0^2 - \omega_f^2} \cdot \sin \omega_f t.$$

Calculate the value of constants:

the value of static strain $\delta_{st} = \dfrac{g \cdot m_D}{c} \approx \dfrac{10 \cdot 20}{3 \cdot 10^4} = 0{,}67 \cdot 10^{-2}$ m,

the value of circular frequency $\omega_0 = \sqrt{\dfrac{c}{m_D}} = \sqrt{\dfrac{3 \cdot 10^4}{20}} = 38{,}7$ *rad/s*,

$$h_f = \frac{c \cdot a_\xi}{m_D} = \frac{3 \cdot 10^4 \cdot 0{,}5 \cdot 10^{-2}}{20} = 7{,}5 \frac{\text{m}}{\text{s}^2},$$

$$\frac{h_f}{\omega_0^2 - \omega_f^2} = \frac{7{,}5}{\frac{3 \cdot 10^4}{20} - 30^2} = \frac{7{,}5}{1500 - 900} = 1{,}25 \cdot 10^{-2} \text{m},$$

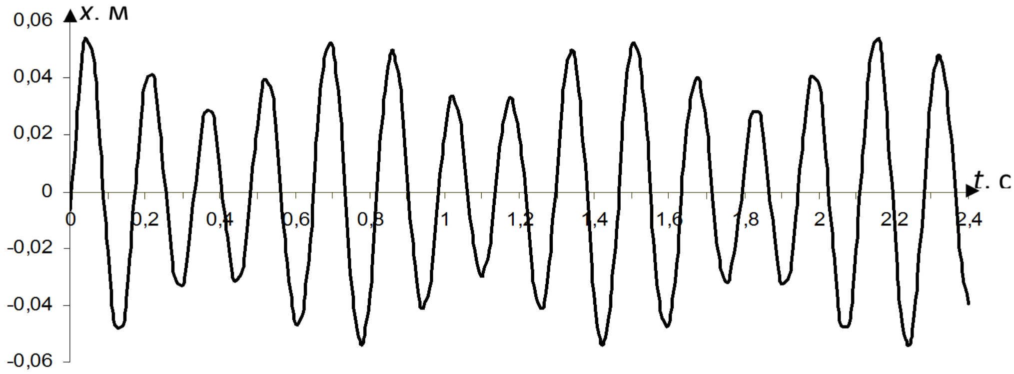
$$\frac{1}{\omega_0} \cdot (V_0 - \frac{h_f \cdot \omega_f}{\omega_0^2 - \omega_f^2}) = \frac{1}{38{,}7} \cdot (2 - \frac{7{,}5 \cdot 30}{\frac{3 \cdot 10^4}{20} - 30^2}) = \frac{1}{38{,}7} \cdot (2 - 0{,}375) = 4{,}2 \cdot 10^{-2} \text{m}.$$

Then the load equation is:

$$x = -0{,}67 \cdot 10^{-2} \cdot \cos 38{,}7\text{t} + 4{,}2 \cdot 10^{-2} \cdot \sin 38{,}7\text{t} + 1{,}25 \cdot 10^{-2} \cdot \sin 30\text{t}\,(\text{m}).$$

Checking: If $t = 0$ we get $\text{x}\big|_{\text{t}=0} = -0{,}67 \cdot 10^{-2}$ m, that coincides with the value of the earlier defined initial coordinate x_0.

Answ. The load has kinematics oscillations by law $x = -0{,}67 \cdot 10^{-2} \cdot \cos 38{,}7\text{t} + 4{,}2 \cdot 10^{-2} \cdot \sin 38{,}7\text{t} + 1{,}25 \cdot 10^{-2} \cdot \sin 30\text{t}$ (м), with free circular frequency ω_0= 38,7 rad / s and exciting frequency ω_f=30 rad / s. The respective periods are T_0= 0,16 s and T_f= 0,21 s. The diagram of ratio the coordinate *x* of time *t* is



1.5 The Main Principles of Dynamics

Measures of particle system motion: *total linear momentum, total angular momentum and total kinetic energy of the system.*

Measure of particle motion	Measure of particle motion	Measure of particle system motion	Effect of forces	General principle
Linear momentum (vector, $\left[kg\cdot\frac{m}{s}\right]$)	$\vec{q}=m\vec{v}$	$\vec{Q}=\sum_{k=1}^{n}\vec{q}_k$	Total vector of external forces	Force-Linear momentum principle
Angular momentum (vector, $\left[kg\cdot\frac{m^2}{s}\right]$)	$\vec{l}_O=\vec{r}\times m\vec{v}$	$\vec{L}_O=\sum_{k=1}^{n}\vec{r}\times\vec{q}_k$	Total moment of external forces about the center O	Moment-Angular momentum principle
Kinetic energy (scalar, $[J]$)	$T=\frac{mv^2}{2}$	$T=\sum_{k=1}^{n}\frac{m_k v_k^2}{2}$	Total work done by external and internal forces	Work-Energy principle

1.5.1 A particle linear momentum principle

A **particle linear momentum** is a vector value, which is equal to the multiplication of its mass and its vector velocity

$$\overline{Q}=m\overline{v}.$$

Linear momentum direction is the same as that of the velocity.

Elementary linear impulse of the force $\overline{F}$ is the multiplication of force and elementary time defined as $d\overline{S}=\overline{F}dt$.

Total linear impulse of the force of time period $\boldsymbol{\tau}$ is the vector

$$\overline{S} = \int_0^{\tau} \overline{F} dt.$$

The projection of the *linear impulse* on the coordinate axis:

$$S_x = \int_0^{\tau} F_x dt; \quad S_y = \int_0^{\tau} F_y dt; \quad S_z = \int_0^{\tau} F_z dt.$$

Force-linear momentum principle for a particle

Differential form. The resultant of all forces acting on a particle equals the first derivative of the linear momentum in time**.**

$$\frac{d\overline{Q}}{dt} = \frac{d(m\overline{v})}{dt} = \overline{R}.$$

The projection of this principle on the coordinate axis:

$$\frac{d(mv_x)}{dt} = R_x; \quad \frac{d(mv_y)}{dt} = R_y; \quad \frac{d(mv_z)}{dt} = R_z.$$

Integral form. Change the linear momentum of a material point over a period of time $\tau = t_1 - t_0$ is equal to the impulse of force resultant over the same period of time:

$$m\overline{v}_1 - m\overline{v}_0 = \overline{S},$$

where $\overline{v}_0, \overline{v}_1$ – velocity of the point in time t_0 and t_1 respectively.

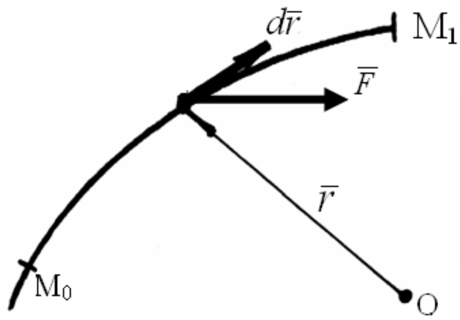
In the projections of this principle on the coordinate axis:

$$m v_{1x} - m v_{0x} = S_x; \; m v_{1y} - m v_{0y} = S_y; \; m v_{1z} - m v_{0z} = S_z.$$

1.5.2 The Work-Energy principle

Kinetic energy of the material point is scalar value equals a half of product mass of point on the square of its velocity: $T = \frac{1}{2} m \overline{v}^2.$

Elementary work $d'A$ of the force $\overline{F}$ on elementary displacement



(infinitely small) of the force point application $d\overline{r}$ is a scalar value:

$$d'A = \overline{F} \cdot d\overline{r} = Fdr\cos(\overline{F}, \hat{\ } d\overline{r}).$$

If under the applied force $\overline{F}$ the material point moves from position M_0 to M_1, then the work of force $\overline{F}$ on the trajectory $M_0 M_1$ is equal to

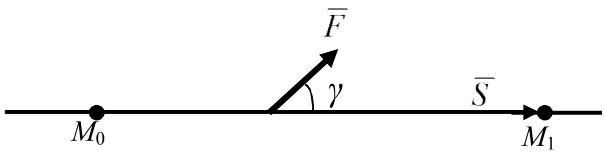
$$A = \int_{M_0 M_1} d'A = \int_{M_0 M_1} \overline{F} \cdot d\overline{r} = \int_{M_0 M_1} F\cos(\overline{F}, \hat{\ } d\overline{r})dr.$$

Formulas for determining the elementary and total work of force in an analytical form:

$$d'A = F_x dx + F_y dy + F_z dz;$$
$$A = \int_{M_0 M_1} (F_x dx + F_y dy + F_z dz).$$

The work of *constant force* $\overline{F}$ moving along a straight-line $\overline{S}$ is defined by the formula

$A = FS\cos(\overline{F}, \hat{\ } \overline{S}) = F \cdot S \cdot \cos\gamma.$

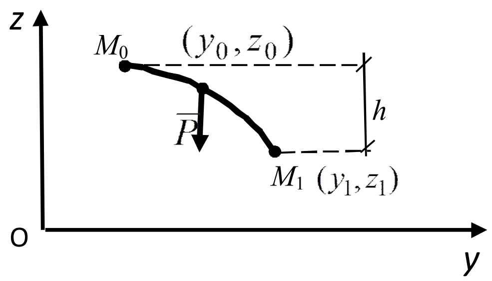


The unit of *work* is the Joule (J). Dimension of work [A]= N·m=J.

***Work of a weight.* (*P* = *mg*):**

$A = Ph$,

where $h = z_0 - z_1$ - the vertical displacement from the initial and final points position. *Notice* that the work done by gravity depends only on the vertical movement of the object.



Work of a spring force the elongation of the spring from a position x_1 to a position x_2: $A = \frac{c}{2}(x_1^2 - x_2^2).$

Consider spring force is exerted to a horizontal force $F_x = (-cx)$ that is proportional to its deflection in the ***x*** direction independent on how a body moves.

$$A = \int_0^t F \cdot v dt = -\int_0^t cxv_x dt = -\frac{c}{2}(x_1^2 - x_2^2)$$

Work of a friction force. $F_{fr} = fN$; $A = -F_{fr} d\ cos\varphi$. *Note* that for the *friction force* ($\overline{F}_{fr} \times \overline{d} < 0$ - *always*) and the velocity of the object is always reduced!

The work applied by the force F ***is zero*** if:

s = 0: displacement equal to zero,

cos $\boldsymbol{90^0 = 0}$: force perpendicular to displacement.

The Work-Energy principle

Differential form. The differential of the kinetic energy of a material point equals the elementary work of forces applied to a point

$$dT = d(mv^2/2) = d'A$$

Integral form. The change in the point's kinetic energy *KE* equals the work *A* of applied resultant of external forces to a point: *difference of the particle's final and initial kinetic energy, respectively, equals the sum of the work done by all the forces acting on the particle as the particle moves from point 1 to point 2.*

$$\frac{m v_1^2}{2} - \frac{m v_0^2}{2} = A,$$

where v_i and v_f are the velocities of the particle before and after the application of force, ***m*** is the point's mass, $A = \sum_{i=1}^{n} A_i$ is the algebraic sum of the all forces work applied to a point on the trajectory.

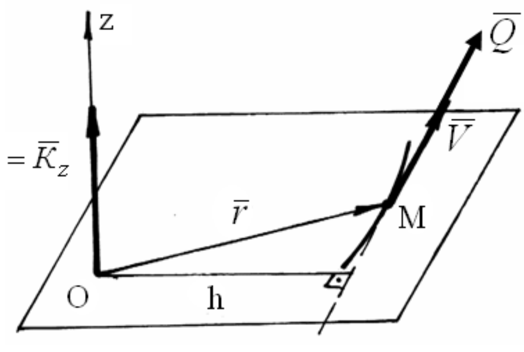
1.5.3. Moment-Angular momentum principle

Angular momentum of a material point about the center *O* is vector product of vector-position, the start of which is at the point *O*, on the *linear momentum* this point

$$\overline{K}_0 = \overline{M}_0(\overline{Q}) = \bar{r} \times \overline{Q} = \bar{r} \times m\bar{v}.$$

***Angular momentum of a material point about axis Oz*:**

$$K_z = M_z(\overline{Q}) = \pm m v\, h.$$



Moment-Angular momentum principle for the point: the derivative in time from the total angular momentum about the point *O* equals the total moment $M_O(R)$ of external forces acting on the point about the same fixed point O.

$$\frac{d\overline{K}_0}{dt} = \overline{M}_0(\overline{R}).$$

In the projections of this principle on the coordinate axis:

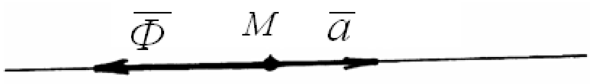
$$\frac{dK_x}{dt} = \sum_{i=1}^{n} M_x(\overline{F}_i); \; \frac{dK_y}{dt} = \sum_{i=1}^{n} M_y(\overline{F}_i); \; \frac{dK_z}{dt} = \sum_{i=1}^{n} M_z(\overline{F}_i).$$

So *the derivative in time from the total angular momentum about the fixed axis equals the total moment $M_{Oz}(R)$ of external forces acting on the point about the same fixed axis O.*

1.6 D'Alambert's principle for the point

Inertial force of material point $\overline{\Phi}$ is equal to the product of the mass of a point on the magnitude of its acceleration and is directed oppositely to the acceleration point vector

$$\overline{\Phi} = -m\overline{a}; \; \Phi = ma.$$



If the point moves on the curvature trajectory, the force of inertia can be given as the sum of two components:

$$\overline{\Phi} = \overline{\Phi}_n + \overline{\Phi}_\tau,$$

where
$$\overline{\Phi}_n = -m\overline{a}_n; \; \Phi_n = m\frac{v^2}{\rho};$$

$$\overline{\Phi}_\tau = -m\overline{a}_\tau; \; \Phi_\tau = m\frac{dv}{dt}.$$

D'Alambert's principle *for the constrained material point*: when the material point is moving, the active forces and reactions of constraints, and the force of inertia conventionally applied of the material point, this system represent a balance of forces

$$\overline{F} + \overline{R} + \overline{\Phi} = 0,$$

where $\overline{F}$ – is total vector of active forces; $\overline{R}$ – is total vector of external reactions of constraints; $\overline{\Phi}$ – inertial force of point.

2 DYNAMICS OF MECHANICAL SYSTEM AND RIGID BODY

Mechanical system (material system) is a set of material points, the position and motion of each depend on the position and motion of the other.

The forces acting on the mechanical system are divided into external and internal.

External forces $\overline{F}^e$ are the forces of interaction of a mechanical system points with bodies which do not belong to this system.

Internal forces $\overline{F}^i$ are the forces of interaction between points belonging to mechanical system.

Internal forces properties:

1) the *resultant vector* of the internal forces of the system is zero: $\overline{F}^i = 0$.

2) the *resultant moment* of the internal forces of the system about any point is zero: $\overline{M}_0(\overline{F}^i) = 0$.

The mass of the mechanical system is equal to the sum of the masses of all points of the system:

$$M = \sum_{i=1}^{n} m_i .$$

The center of mass is the point at which all of the mass in a system is concentrated. The center of mass of a mechanical system is a geometric point which vector-position is determined by expression:

$$\overline{r}_c = \frac{\sum_{i=1}^{n} m_i \overline{r}_i}{\sum_{i=1}^{n} m_i} .$$

Coordinates of the center of mass is determined by expression:

$$x_c = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}, \quad y_c = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}, \quad z_c = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i} .$$

The center of mass is a geometric (not material) point by definition. The center of gravity of the system coincides with the center of mass.

2.1 Mass-Center motion of system principle

Principle: Mass-center of mechanical system moves as point acted on by force. Mass of the point is equal to mass of the system. The force is equal to the total vector of external forces applied to the system.

$$M\overline{a}_c = \overline{F}^e.$$

Differential equations of the center of mass motion in the projections on the axis of the Cartesian coordinate system:

$$M\ddot{x}_C = \sum_{i=1}^{n} F_{ix}^e; \quad M\ddot{y}_C = \sum_{i=1}^{n} F_{iy}^e; \quad M\ddot{z}_C = \sum_{i=1}^{n} F_{iz}^e.$$

These equations are differential equations of the translational motion of a absolutely rigid body.

Internal forces do not enter into these equations and do not directly affect the motion of the center of mass of the mechanical system. In a changing system, internal forces cause motion of the system points; change their relative position, without changing the position of the center of mass.

Internal forces are cause of external forces which make the center of mass move. For example, in vehicles (tram, trolley, car, etc.), the internal forces of the engine influence their motion through the friction forces between the driving wheels and the support surface (rails, track).

The consequences of the theorem (*principle of conservation of motion* of the center of mass):

1) if $\overline{F}^e = 0$, then $\overline{a}_c = 0$ i $\overline{v}_c = const$ – the center of mass of the system moves uniformly and rectilinearly or is at rest.

2) if $\overline{F}_x^e = \sum_{i=1}^{n} F_{ix}^e = 0$, then $a_{cx} = \ddot{x}_c = 0$ and $v = \dot{x}_c = const$ – the projection of the velocity of the center of mass on the axis x is constant.

2.2 Force-Linear momentum principle for a mechanical system

The *total linear momentum* of a mechanical system is a vector equal to the geometric sum of the *linear momentum* of all material points of the system.

$\overline{Q}$ is *total linear momentum* of the mechanical system

$$\overline{Q} = \sum_{i=1}^{n} m_i \overline{v}_i.$$

The *principle linear momentum* of a mechanical system is equal to the product of the mass of the system on the velocity vector of its center of mass:

$$\overline{Q} = M \cdot \overline{v}_c.$$

Force-momentum principle for a mechanical system. Differential form. The total external force on a mechanical system equals the derivative of time of the *principle* linear momentum of the mechanical system.

$$\frac{d\vec{Q}}{dt} = \vec{F}^{(e)}$$

The scalar view of equation is

$$\frac{dQ_x}{dt} = \sum_{i=1}^{n} F_{ix}^{e}; \quad \frac{dQ_y}{dt} = \sum_{i=1}^{n} F_{iy}^{e}; \quad \frac{dQ_z}{dt} = \sum_{i=1}^{n} F_{iz}^{e}.$$

Integral form. The principle linear impulse of force acting on a mechanical system equals the corresponding change in linear momentum of the system of time $t_1 - t_0 = \tau$

$$\overline{Q}_1 - \overline{Q}_0 = \overline{S}^e,$$

where $\overline{Q}_1$ and $\overline{Q}_0$ – the *linear momentum* of the system at finite t_1 and initial t_0 moments of time;

$\overline{S}^e = \int_0^{\tau} \overline{F}^e dt$ – The total *linear impulse* of external forces in time $t_1 - t_0 = \tau$.

Two Corollary of the Force-momentum principle for mechanical system (*principle of conservation of linear momentum*)*:*

1) when the total vector of external forces is zero $\sum \overline{F}_k^e = 0$, then $\frac{d\overline{Q}}{dt} = 0;\ \overline{Q} = Const$.

If the total external force on the mechanical system during a time interval equals zero than the principle linear momentum of the system is unchanged during the time interval.

2) $\sum \overline{F}_{kx}^e = 0$, then $\frac{dQ_x}{dt} = 0 \rightarrow V_x = 0 \rightarrow Q_x = const$.

If the projection on an axis of the total external force on the mechanical system during a time interval equals zero than the projection on the axis of the principle linear-momentum of the system is unchanged during the time interval.

2.3 Principle of angular impulse and momentum

The *angular momentum of a particle of mass m about the point O of the inertial reference* is given by the vector multiplication

$$\overline{K}_o = \overrightarrow{M}_o(\vec{q}) = \vec{r} \times m\vec{v} .$$

The ***total angular momentum about point O fixed in an inertial reference*** is the vector sum of the angular momenta of the points of the mechanical system

$$\overline{K}_0 = \sum_{i=1}^{n} \overline{K}_{ox} = \sum_{i=1}^{n} \overline{r}_\kappa \times \overline{Q}_\kappa = \sum_{i=1}^{n} \overline{r}_\kappa \times m_\kappa \overline{V}_\kappa .$$

The ***total angular momentum*** about axis *Oz* is the sum of the angular momenta of the points of the mechanical system about given axis:

$$\overline{K}_0 = K_x \bar{i} + K_y \bar{j} + K_z \bar{k} \qquad \text{or} \qquad K_z = \sum_{i=1}^{n} K_{oz} .$$

Principle: *The total moment* $\overline{M}_O^{(e)}$ *of external forces acting on a mechanical system about a point O fixed in an inertial reference equals the first derivative on the time of the total angular momentum relative to the point O.*

$$\frac{d\overline{K}_0}{dt} = \overline{M}_0^e ,$$

where $\overline{M}_0^e = \sum_{i=1}^{n} \overline{M}_0(\overline{F}_i^e)$ is t*he total moment* $\overline{M}_O^{(e)}$ *of external forces about point O.*

Vector equation in projections on the Cartesian coordinate system:

$$\frac{dK_x}{dt} = \sum_{i=1}^{n} M_x(\overline{F}^e); \quad \frac{dK_y}{dt} = \sum_{i=1}^{n} M_y(\overline{F}^e); \quad \frac{dK_z}{dt} = \sum_{i=1}^{n} M_z(\overline{F}^e).$$

Principle of conservation of *angular momentum*:

1) if the total moment of external forces $\overline{M_O}^{(e)} = 0$ acting on the mechanical system about a point *O* has no component in any direction, the total angular momentum about the point *O* remains *constant:* $\overline{M}_0^e = 0$, then $\overline{K}_0 = const$.

2) if the total moment of external forces acting on a the mechanical system about an axis ***x*** $M_x^{(e)} = 0$ *equals zero* during finite time interval, the total angular momentum about the axis remains *constant* during this time interval.

$$\sum_{i=1}^{n} M_z(\overline{F}_i^e) = 0 \text{, then } K_z = const .$$

2.3.1 The differential equation of the rigid body rotation about the axis Oz:

$$I_z \frac{d\omega}{dt} = \sum_{i=1}^{n} M_z(\overline{F}_i^e),$$

where I_z – moment of inertia of the rigid body about axis z; ω – angular velocity body about axis z; $M_z(\overline{F}_i^e)$ – moment of external forces about the axis Oz.

2.3.2 Moments of inertia of a rigid body (mechanical system)

Moment of inertia about axis (*the second moment*) is the sum of production mass of points on its squared distances from the axis:

$$I_z = \sum_{i=1}^{n} m_i h_i^2 .$$

Moment of inertia of rigid body:

$$I_z = \lim_{n\to\infty} \sum_{i=1}^{n} h_{iz}^2 \Delta m_i = \int_{(m)} h_z^2 dm \ ,$$

where $h_{iz}(h_z)$ – distance from axis of body's point mass $\Delta m_i(dm)$; x, y, z – coordinates of body's point.

Axial moments of body inertia:

$$I_x = \int_{(m)} (y^2 + z^2)dm; \quad I_y = \int_{(m)} (x^2 + z^2)dm; \quad I_z = \int_{(m)} (y^2 + x^2)dm .$$

Polar moment of inertia I_0 (about the pole *O*) and planar $I_{xoy}, I_{xoz}, I_{yoz}$ (about the plate) moment of inertia:

$$I_0 = \int_{(m)} r^2 dm; \ I_{xoy} = \int_{(m)} z^2 dm; \ I_{xoz} = \int_{(m)} y^2 dm; \ I_{yoz} = \int_{(m)} x^2 dm ,$$

where r – distance the point from the pole *O*.

Polar moment of inertia equals a half of the sum of inertia of axis moments and the sum inertia of planar moments:

$$I_0 = I_{xoy} + I_{xoz} + I_{yoz} = \frac{1}{2}(I_x + I_y + I_z)$$

The product of inertia of a plane area are value, which expressed by formulas:

$$I_{xy} = \int_{(m)} xydm; \ I_{xz} = \int_{(m)} xzdm; \ I_{yz} = \int_{(m)} yzdm.$$

Products of inertia may be positive, negative, or zero, depending on the position of the *xy* axes with respect to the area. If the product of inertia of an area is zero with respect to any pair of axes this axis *x, y* called *the centroidal axis* in point *O*. If this point coincide with mas-center of the body, then coordinate axis are *main centroidal axis of inertia*.

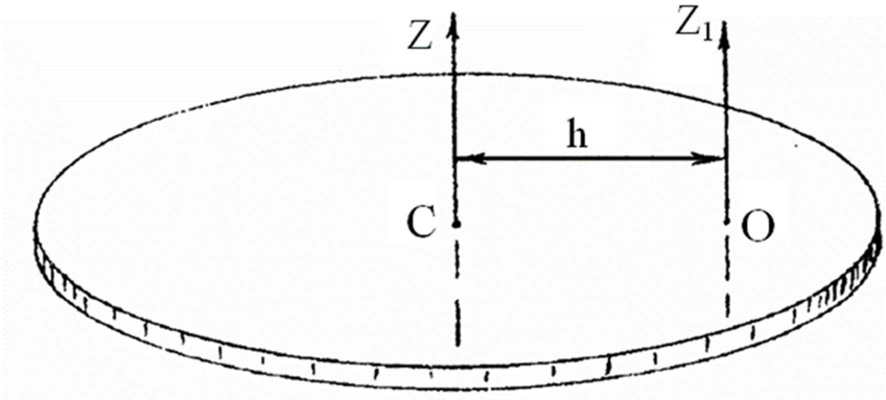
Radius of gyration ρ_z of the body about axis *z* is the value, which equals the distance from the axis to the material point. Which value is determined by expression:

$$I_z = m\rho_z^2; \quad \rho_z = \sqrt{I_z / m}.$$

Parallel-axis theorem for moments of inertia: *The moment of inertia of an area with respect to any axis in its plane is equal to the moment of inertia with respect to a parallel centroidal axis plus the product of the area and the square of the distance between the two axes.*

$$I_{z_1} = I_{C_z} + Mh^2,$$

where z_1 – arbitrary axis; *z* – the axis passing through the mass-center of *C* of the body parallel to the axis z_1.

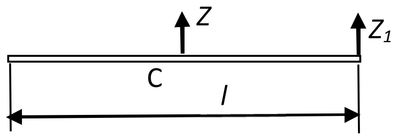


Moment of some elements inertia

Uniform rod

$I_{C_z} = Ml^2/12; \; I_{z_1} = Ml^2/3,$

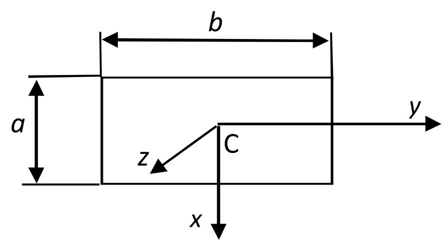
where *M* – mass of body.



Rectangular area

$I_{C_x} = Mb^2/12; \; I_{C_y} = Ma^2/12.$

$$I_{C_z} = \frac{M}{12}(a^2 + b^2).$$



Hollow cylinder:

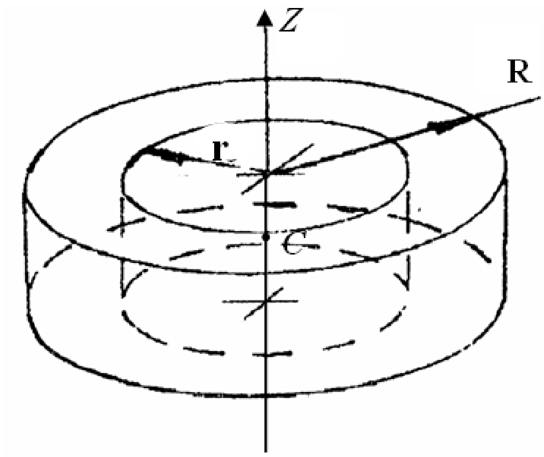
$$I_{C_z} = \frac{1}{2} M (R^2 + r^2).$$

Thin ring:

$(r \to R)$: $I_z = MR^2$.

Cylinder (circular disk)

$(r = 0)$: $\quad I_{C_z} = MR^2 / 2.$



2.4 The Work-Energy principle for mechanical system

Kinetic energy of *mechanical system* is a *scalar* value, which equals the sum of *kinetic energy* material points and has units of joules (*J*) and *Nm*.

$$T = \sum_{i=1}^{n} \frac{m_i v_i^2}{2},$$

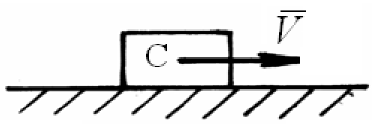
where m_i – mass of the point, v_i – velocity of the point.

Koenig's Theorem: The kinetic energy of a system is equal to the sum of the kinetic energy of the center of mass of the system and the kinetic energy of the system in its relative motion with respect to the coordinate system, which moves translationally with the center of mass:

$$T = \frac{M v_c^2}{2} + \sum_{i=1}^{n} \frac{m_i v_{ri}^2}{2},$$

where m_i – mass of the point of the system; v_{ri} – relative point velocity about mass-center; M – mass of mechanical system; v_c – velocity of mass-center of system. The kinetic energy of a rigid body determined by the formula:

for *translation* motion $T = \dfrac{Mv_c^2}{2}$;

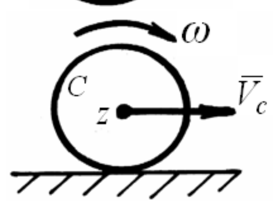


for *rotation* about a fixed axis

$$T = \frac{1}{2} I_z \omega^2;$$

for *general plane* motion

$$T = \frac{M v_c^2}{2} + \frac{1}{2} I_{C_z} \omega^2,$$



where M – mass of body, v_c – velocity of mass-center of system; I_z, I_{c_z} – moment of inertia about axis *z* (axis of rotation) or axis *z*, which pass through mass-center; ω

– angular velocity of body.

The work-energy principle for mechanical system:

–*differential form*: the differential kinetic energy of the mechanical system is equal to the sum of elementary works of external and internal forces applied to the points of the system:

$$dT = \sum_{i=1}^{n} d'A_i^e + \sum_{i=1}^{n} d'A_i^i;$$

– *integral form*: the change in the kinetic energy of the mechanical system on a certain displacement is equal to the sum of the work of external and internal forces on the same displacement:

$$T_1 - T_0 = \sum_{i=1}^{n} A_i^e + \sum_{i=1}^{n} A_i^i,$$

where T_1 and T_0 – the kinetic energy of the system at the end and the start of the way; $\sum A_i^e$ – sum of works of external forces; $\sum A_i^i$ – sum of works of internal forces.

The work of the force applied to a rigid body rotating about a fixed axis z is equal to the product of the moment of force relative to a given axis by the angle of rotation of the body:

$$d'A = M_z(\bar{F})d\varphi; \;\; A = \int_0^{\phi} M_z(\bar{F})d\varphi,$$

where $M_z(\bar{F})$ – moment of force about axis; $d\varphi, \varphi$ – elementary and real angle of rotation of the body.

If $(M_z = const)$ we get $A = M_z(\bar{F})\varphi.$

Elementary work of forces applied to a free-moving rigid body is equal to the sum of the work of the principal vector $\bar{F}$ of the mechanical system on the elementary displacement $d\bar{r}_0$ of the pole O and the work of the principal moment $\bar{M}_O$ of this system of forces with the respect to the pole on the elementary rotation movement:

$$d'A = \bar{F} \cdot d\bar{r}_0 + \bar{M}_0 \cdot d\bar{\varphi}.$$

Power N is a physical quantity, which characterizes the velocity of work.

$$N = \bar{F} \cdot \bar{V} = F_\tau \cdot V = M_z \cdot \omega,$$

where $\bar{V}$ – velocity of the point, F_τ – projection of force on tangent axis, $M_z(\bar{F})$ – moment of force about axis of rotation , ω – angular velocity of body. Units of *power* is *Watt* (*W*) ($1W = 1N \times m/s = 1J/s$).

2.5 D'Alambert's principle for mechanical system

D'Alambert's principle for mechanical system: At any time, the sum of resultants of active and reactive forces acting on the system and inertial force is zero. At any time, the sum of principle moments of active, external reactive and inertial forces acting on the moving system about any point *O* is zero:

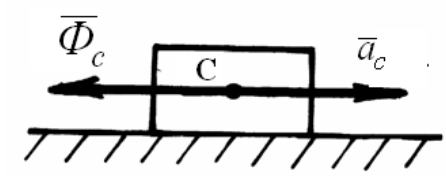
$$\sum_{i=1}^{n}\bar{F}_i + \sum_{i=1}^{n}\bar{R}_i + \sum_{i=1}^{n}\bar{\Phi}_i = 0; \quad \sum_{i=1}^{n}\bar{M}_0(\bar{F}_i) + \sum_{i=1}^{n}\bar{M}_0(\bar{R}_i) + \sum_{i=1}^{n}M_0(\bar{\Phi}_i) = 0.$$

The forces of inertia of the body are given:

- if *translation motion* to resultants force of inertia $\bar{\Phi}_C$:

$\bar{\Phi}_C = -M\bar{a}_C$,

which applied to mass-center *C*;

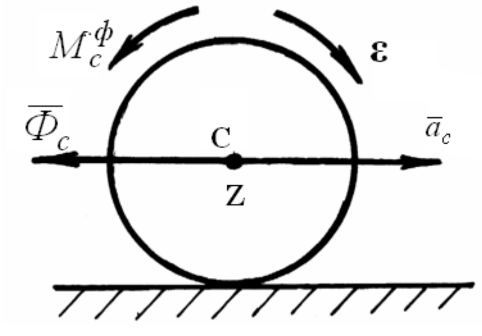


- in *rotation* about fixed axis passing through the mass-center of the body – to *principle moment* $\bar{M}_z^{in}$

$$\bar{M}_z^{in} = -I_z\bar{\varepsilon}; \quad \bar{\Phi}_O = -M\bar{a}_C;$$

- in *general plane motion* to the principal vector of inertial forces $\bar{\Phi}_C$ applied to the mass-center and the principal moment of inertial forces $\bar{M}_{c_z}^{in}$:

$$\Phi_C = -M\bar{a}_C; \quad \bar{M}_{Cz}^{in} = -I_{C_z}\bar{\varepsilon},$$



where M – mass of the body; $\bar{a}_C$ – acceleration of the mass-center; $\bar{\varepsilon}$ – angular acceleration; I_z, I_{c_z} – moments of inertia rotating body about fixed axis z and axis z_C, which pass through mass-center *C*.

3 ELEMENTS OF ANALYTICAL MECHANICS
3.1 Classification of constraints

There are free and non-free mechanical systems. A mechanical system is called *free* if the movement of its points is unrestricted by any bodies (constrains). If the movement of the mechanical system is restricted by the constrains, then it is called non-free.

Classification of Constraints. The restrictions which constraints put on a mechanical system are expressed analytically in the relations (equations or inequalities) between the time, coordinates, and velocities of the points belonging to the mechanical system.

Classification of Constraints:

Geometric (finite) are constraints, which do not include point velocities of systems:

$$f(t, x_i, y_i, z_i) = 0, \qquad (i = 1, \dots n).$$

Kinematic (differential) are such constraints which include the velocities of system points:

$$f(t, x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i) = 0, \qquad (i = 1, \dots n).$$

Holonomic (integrated) constraints can be expressed as a function of the coordinates and time. **Non-holonomic** (non-integrated): constraint relations which are not holonomic.

Conservative: total mechanical energy of the system is conserved while performing the constrained motion. Constraint forces do not do any work. **Dissipative**: constraint forces do work and total mechanical energy is not conserved.

Scleronomic: constraint relations do not explicitly depend on time,

Rheonomic: constraint relations depend explicitly on time,

Bilateral (restraining): at any point on the constraint surface both the forward and backward motions are possible. Constraint relations are not in the form of inequalities but are in the form of equations:

$$f(x, y, z, \dot{x}, \dot{y}, \dot{z}) = 0$$

Unilateral (non-restraining): at some points no forward motion is possible. Constraint relations are expressed in the form of inequalities:

$$f(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \le 0.$$

3.2 Virtual work principle

Virtual displacement. The displacement is called **virtual (***infinitesimal)* **displacement** of a system, which is allowed by the system's constraints in this time (in this system position).

Ideal constraints are those which **the virtual work** done by the constraint forces along the virtual displacement must be zero:

$$\sum_{i=1}^{n} \overline{R}_i \cdot \delta \overline{r}_i = 0 .$$

Principle of virtual work. The condition for static equilibrium is that the virtual work done by all the applied forces should vanish, provided the virtual work done by all the constraint forces escape.

$$\sum_{i=1}^{n} \delta A_i^a = \sum_{i=1}^{n} \overline{F}_i \cdot \delta \overline{S}_i = 0 .$$

Examples of ideal constraints: ideally smooth surface; ideal joints (bearing), rods, etc.

The principle of virtual work states that in equilibrium the virtual work of the forces applied to a system is zero. This means the virtual work of the constraint forces must be zero as well.

scalar form $\sum_{i=1}^{n} F_i \delta S_i \cos(\overline{F}_i, \hat{\ } \delta \overline{S}_i) = 0$;

analytical form $\sum_{i=1}^{n} (F_{ix} \delta x_i + F_{iy} \delta y_i + F_{iz} \delta z_i) = 0$.

3.3 Generalized coordinate, velocity and generalized force

Generalized coordinates $q_1, q_2, ... q_S$ called such independent parameters, when the task of which can individually determine the position of all points of the system. Such parameters can be Cartesian coordinates, angles, distances, etc. The number S of such independent parameters are called **the number of degrees of freedom** of the mechanical system.

Time derivatives of generalized coordinates, that are $\dot{q}_1, \dot{q}_2, ... \dot{q}_S$, magnitudes, are called **generalized system velocities**.

Let's calculate the work of the points of the active forces system $\overline{F}_k$ applied to the virtual displacement of the system:

$$\delta A = \sum_{k=1}^{n} \overline{F}_k \cdot \delta \overline{r}_k = \sum_{i=1}^{S} \left(\sum_{k=1}^{n} \overline{F}_k \frac{\partial \overline{r}_k}{\partial q_i} \right) \delta q_i = \sum_{i=1}^{S} Q_i \cdot \delta q_i ,$$

where $Q_i = \sum_{k=1}^{n} \overline{F}_k \frac{\partial \overline{r}_k}{\partial q_i} = \sum_{k=1}^{n} \left(F_{kx} \frac{\partial x_k}{\partial q_i} + F_{ky} \frac{\partial y_k}{\partial q_i} + F_{kz} \frac{\partial z_k}{\partial q_i} \right)$, i=1,2, ... S.

Values Q_1, Q_2,...Q_S , which are multipliers in the virtual displacements of generalized coordinates in the formula of the active forces work on the virtual displacement of the system are called **generalized forces**.

To calculate the **generalized force** Q_i (i=1,2, ... S), it is enough to give a virtual displacement of the coordinate q_i and to determine the work δA_i of active forces on the displacements of the system points, which are caused only by a change of the coordinate q_i.

We get $Q_i = \delta A_i / \delta q_i$, ($i$=1,2, ... S).

If the active forces are potential then the generalized forces will be equal to the partial derivative of the *potential energy* U (q_1, q_2,..., q_s) in the generalized coordinates:

$$Q_i = -\frac{\partial U}{\partial q_i}.$$

3.4 Dynamics Equations of the System

General equation. In the motion of a system obeying a holonomic two-sided ideal constraints, the sum of the active and inertial forces on any virtual displacement of the system must be zero:

$$\sum_{k=1}^{n} \bar{F}_k \cdot \bar{\delta} q_k + \sum_{k=1}^{n} \bar{\Phi}_k \cdot \bar{\delta} q_k = 0$$

or $$Q_i + Q_i^{in} = 0 \quad (i = 1,2, \ ..., S),$$

where $\bar{\Phi}_k$ – inertial force of the point; Q_i^{in} – generalized force of inertia; δq_k – virtual displacement of the point.

Lagrange's Equations of the second kind is the following:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \quad (i\text{=1,2, ...}, S),$$

where T – kinetic energy of the system, presented as a function of generalized coordinates, generalized velocities and time; $\frac{\partial T}{\partial \dot{q}_i}$, $\frac{\partial T}{\partial q_i}$ – partial derivatives of kinetic energy on generalized velocities and coordinates.

3.5 Impact

Impact occurs when two bodies collide with each other during a very *short* period of time, causing relatively large (impulsive) forces to be exerted between the bodies.

Impact – the phenomenon at which the velocities of body points in a very small (close to zero) time interval change to a final value.

Impact forces are the forces at which an impact occurs. Impact forces operate for a very short period of time and reach very large values.

In impact theory, the momentum are considered (not as impact forces) but as a measure of interaction.

Impact impulse – a vector, which i determined by the formula

$$\bar{S}^{im} = \int_0^{\tau} \bar{F}^{im} dt,$$

where τ – time of impact.

Force-Linear momentum principle for material point upon impact (general equation of impact theory of material point): the change in the linear momentum of a material point during impact is equal to the geometric sum of the impact impulses acting on the point:

$$m(\bar{U} - \bar{V}) = \sum_{i=1}^{n} \bar{S}_i^{im},$$

where $\bar{U}$, $\bar{V}$ – the velocities of the point after and before the collision.

Coefficient of restitution in impact:

$$k = \frac{\left(U_n^r\right)}{\left(V_n^r\right)}, \qquad 0 \le k \le 1,$$

where $\left(U_n^r\right)$, $\left(V_n^r\right)$ – the magnitudes of the normal components of the relative contact point velocity of bodies after and before impact, respectively.

Physical meaning of two limits:

1) $k = 1$ – *Elastic Impact* (the bodies after the impact have different velocities);

2) $k = 0$ – *Plastic Impact* (after collision both particles couple or stick *together* and move with a common velocity).

Force-Linear momentum principle for material system upon impact: the change in the linear momentum of a mechanical system during impact is equal to the geometric sum of the impact impulses acting on the system:

$$\bar{Q}_1 - \bar{Q}_0 = \sum_{i=1}^{n} \bar{S}_i^{im},$$

where $\bar{Q}_1, \bar{Q}_0$ – the linear momentum of a mechanical system after and before the collision.

Moment-Angular Momentum principle for material system upon impact: change in the *total angular momentum* of a mechanical system about fixed pole A

during impact is equal to the geometric sum of the external impact impulses relative to the same pole A:

$$\overline{K}_{A_1} - \overline{K}_{A_0} = \sum_{i=1}^{n} \overline{M}_A(\overline{S}_i^{im}),$$

where $\overline{K}_{A_1}, \overline{K}_{A_0}$ – total angular momentum of a mechanical system about fixed pole A after and before the collision.

There are two types of impact. *Central impact* occurs when the direction of motion of the mass centers of the two colliding particles is along a line passing through the mass centers of the particles. When the motion of one or both of the particles make an angle with the line of impact, the impact is said to be *oblique impact*.

General equation of central impact:

$$m_1 V_{1x} + m_2 V_{2x} = m_1 U_{1x} + m_2 U_{2x}; \quad k = \frac{U_{2x} - U_{1x}}{V_{1x} - V_{2x}},$$

where V_{1x}, V_{2x} – projections of the velocity of the bodies before the impact on the x-axis coinciding with the line of impact; U_{1x}, U_{2x} – projections of the velocity of the bodies after the impact; m_1, m_2 – bodies masses ; K – coefficient of restitution.

Ostrogradskiy-Carno theorem:

$$T_1 - T_0 = \frac{1-k}{1+k}\left(\frac{1}{2}m_1(V_{1x} - U_{1x})^2 + \frac{1}{2}m_2(V_{2x} - U_{2x})^2\right);$$

where $T_0 - T_1$ – change of kinetic energy of two bodies when at impact; T_0, T_1 – kinetic energy system before and after impact; $(V_{1x} - U_{1x})$, $(V_{2x} - U_{2x})$ – velocity change.

In case of **oblique impact** the equation

$$m_1 V_{1n} + m_2 V_{2n} = m_1 U_{1n} + m_2 U_{2n}; \; k = \frac{U_{2n} - U_{1n}}{V_{1n} - V_{2n}},$$

where V_{1n}, V_{2n} and U_{1n}, U_{2n} – projection of body's velocities on the n axis, passing through the mass-centers of these bodies before and after the impact.

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APPENDIX A

Output dates for work "OSCILLATIONS OF A MATERIAL POINT"

In problems, the oscillations of the load *D* or the system of loads *D* and *E*. Find equations of motion of load or system *D* and *E*. Consider the motion vertically, relatively to the *Ox* axis. Combine the reference with the equilibrium of the load *D* or the system *D* and *E*.

1. Load *D* (m_D =2 kg) is attached to the beam *AB*, attached to two identical parallel springs, each spring stiffness and resistance coefficient is c =3 *N/cm*, b = 6 *Ns/cm*. The point of attachment of the load is in the middle of the axes of the spring. At the time t = 0 to the load *D*, which is in the rest position, hang the load *E* (m_E = 1 *kg*).

2. At the time t =0 the rod, which connects the loads *D* (m_D = 1 *kg*) and *E* (m_E = 2 *kg*) cut, and the base begins to make a moving by law ξ=1,5 sin18t (*cm*). Stiffness and resistance coefficients of a spring are c_1 = 12 *N / cm*, c_2 = 36 *N / cm*, $b_1 = b_2 = 0$.

3. At the time t =0 to load *D* (m_D = 0,8 kg) connected to load *E* (m_E = 1,2 *kg*) and give the system of loads *D* and *E* initial velocity v_0 = 0.2 *m/s*, downward. Coefficients of stiffness and resistance is c_1 = 10 *N / cm*, c_2 = 4 *N / cm*, c_3 = 6 *N /cm*, $b_1 = b_2 = 0$.

4. The system loads *D* and *E* are attached to the beam *AB*. The beam connecting to system of two parallel springs. Static deformation of two springs with coefficient of resistance b = 6 *Ns/cm* under the combined action of loads *D* (m_D = 0,5 *kg*) and *E* (= 1,5 *kg*) is δ_{st} = 4 *cm*. At the time t = 0 the rod, which connects the loads, is cut.

5. At the time, to load *D* (m_D = 1,6 *kg*), which attached to a spring with stiffness c = 4 *N/cm* and resistance b = 0 , connected load *E* (m_E = 2,4 *kg*), and at point *B* the base begins to make motion by the law $\xi = 2\ sin5t$ (*cm*).

6. Load *D* has moved without initial velocity on inclined plane () the distance s = 0,1 *m*, at the time t = 0 load *D* (m_D = 4 *kg*) connects with the non-deformed springs which connected in series, and have stiffness and resistance coefficients c_1 = 48 *N / cm*, c_2 = 24 *N / cm*, $b_1 = b_2 = 0$.

7. At the time t = 0 load *D* (m_D = 2 *kg*) added at point *A* without initial velocity to the system non-deformed series-connected springs, which have stiffness and resistance coefficients c_1 = 12 *N / cm*, c_2 = 6 *N / cm*, $b_1 = b_2 = 0$. At the same time base at the point *B* starts moving along the inclined plane ($\alpha = 45^o$) by the law $\xi = 0{,}02\ sin20t$ (*m*).

8. At the time t = 0 the load *D* (m_D = 1,5 *kg*) added to a point *N* to system of the three non-deformed parallel-series connected springs with beam *AB*, which have

coefficients of stiffness and resistance $c_1 = 4$ N / cm, $c_2 = 6$ N / cm, $c_3 = 15$ N / cm, $b_1 = b_2 = b_3 = 0$. At the same time the load B has the velocity of $v_0 = 0{,}5$ m / s in the positive direction of reference coordinates X. Given angle $\alpha = 45^0$.

9. The load D (m_D=1.2 kg), moved distance S = 0.2 m away without the initial velocity on an inclined plane ($\alpha = 30^o$), is connected to a non-deformed spring having coefficients of stiffness and resistance c = 4,8 N / cm, b = 0 . At the same time, the basis at the point B begins to move along the inclined plane by law $\xi = 0{,}03\sin 12t$ (m).

10. At the time $t = 0$ the load D (m_D =1 kg) without initial velocity joins the system coupled with a rod AB with two identical parallel connected non-deformed springs having each coefficients of stiffness and resistance coefficient c = 1.5 N / cm, b = 4 Ns/m. Take the angle $\alpha = 60^o$.

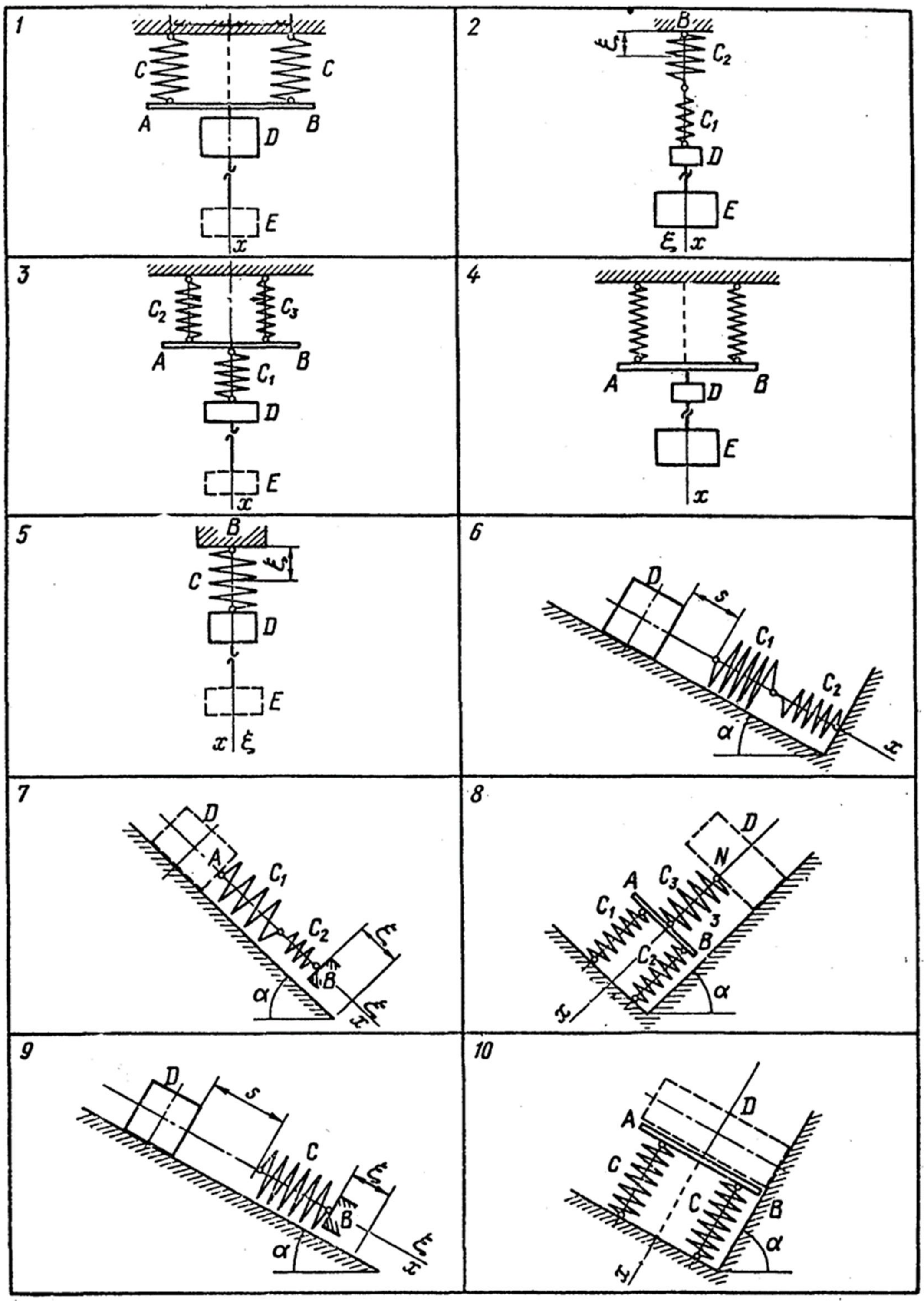
11. Load D (m_D = 2.4 kg) is held at point F to the rod AB. The coefficients of stiffness and resistance of the system of two parallel fixed springs, connected with the rod, $c_1 = 1$ N / sm, $c_2 = 1{,}4$ N / cm, $b = 3$ Ns / m. Before the start of the motion, the load is extend on a value of $\lambda = 2$ cm in the direction of positive reference of the coordinate X at a time t = 0 without initial velocity.

12. At the moment $t = 0$, the load D ($m_D = 3$ kg) is held in a state in which the spring is compressed by a value of 2 cm, released without initial velocity. Coefficients of stiffness and spring resistance $c = 9$ N / cm, $b = 0$. At the same time, the base at point B begins to move according to law $\xi = 1{,}2\sin 8t$ (cm).

13. The load D (m_D =1 kg) (in the equilibrium state shown in the drawing) gives an initial velocity of $v_o = 0.5$ m / s in the direction of positive reference coordinate X. The coefficient of stiffness and spring resistance of three parallel-series connected springs $c_1 = 12$ N / cm, $c_2 = c_3 = c = 3$ N / cm, $b_1 = b_2 = b_3 = 0$.

14. The load D (m_D =1.5 kg) from the equilibrium position (shown in the drawing), before the start of the motion, is deflected by a value of $\lambda = 2.5$ cm in the direction of positive reference X coordinates and released at time $t = 0$, giving the load an initial velocity of $v_o = 0.4$ m / s, directed to the right. The coefficients of stiffness and spring resistance $c_1 = 4.4$ N / cm, $c_2 = 2$ N / cm, $c_3 = 8$ N / cm, $b_1 = b_2 = b_3 = 0$.

15. The load D ($m_D = 1$ kg) is held in equilibrium state, as shown in the drawing, by a system of series-connected springs, which have coefficients of stiffness and resistance $c_1 = 4$ N / cm, $c_2 = 12$ N / cm, $b_1 = b_2 = 0$. At the time $t = 0$, the base at the point B starts to move according to the law $\xi = 1{,}8\sin 12t$ (cm).



Variants 16 – 20.

Find the equation of the oscillatory motion of the load D in the direction of the axis Ox. The rod which connects the springs is weightless. Motion of the block AB, which occupies a state of rest in horizontal position, considered translation. Loads D, E are not separate from each other in combined motion.

16. At the time $t = 0$, a load D ($m_D = 20$ kg) is placed on the stationary load E ($m_E = 10$ kg), giving the loading system D and E an initial velocity of $v_o = 0{,}4$ m / s in the direction of positive reference X. The coefficient of stiffness and resistance of the system bounded by a rod AB of three parallel- series connected springs $c_1 = 200$ N / cm, $c_2 = 160$ N / cm, $c_3 = 140$ N / cm, $b_1 = b_2 = b_3 = 0$. The origin of the reference axis Ox is combine with the position of the static equilibrium of the system of loads D, E.

17. The system of loads D and E set on the spring is in a position of static equilibrium. At time $t = 0$, the load E is removed from the load D. The oscillation frequency of the loads system D and E is $\omega_o = 20$ rad / s, the mass ratio $m_D / m_E = 2 / 3$. The start of the reference axis Ox to combine with the position of the static equilibrium of the load D.

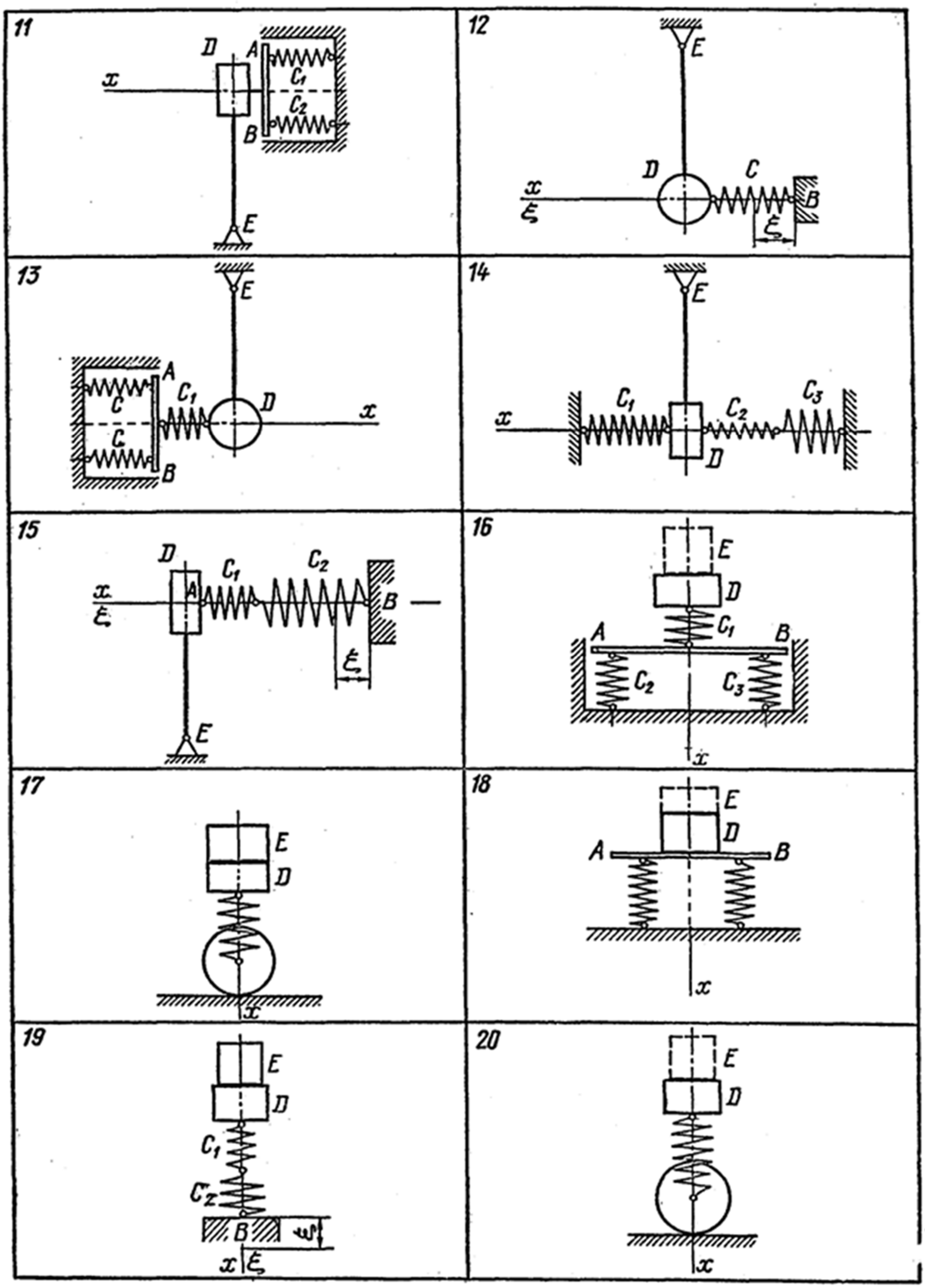
18. The load D is set on the AB beam, which connects the system of two identical parallel-connected springs. At time $t = 0$, on the load D ($m_D = 20$ kg) is set to the top load E ($m_E = 10$ kg). The static deformation of each of the springs with a resistance coefficient $b = 30\sqrt{3}$ $Ns\ m$ under the action of the load D is $\delta_{st} = 2$ cm. The origin of the reference axis Ox to combine with the position of the static equilibrium of the loads system D, E.

19. The system of loads D ($m_D = 15$ kg) and E ($m_E = 25$ kg) set on two series-connected springs which coefficients of stiffness and resistance $c_1 = 250$ N / cm, $c_2 = 375$ N / cm, $b_1 = b_2 = 0$, is in the static equilibrium. In time $t = 0$, the load E is removed from the load D. At the same time, the base at point B begins to move the law $\xi = 0{,}05 \sin 30t$ (m).

20. At the moment $t = 0$ the load E is set to the top of the load D, giving the system of loads D and E an initial velocity $v_o = 0.3$ m / s in the direction of positive reference coordinate X. Oscillation frequencies of load D on the spring of $\omega_o = 24$ rad / s, the mass ratio $m_D / m_E = 3$. Origin the axis Ox combine with the position of the static equilibrium of the loads system D and E.

Variants 21 – 25.

Find the solution of the oscillatory motion of the load D along a smooth inclined plane in the direction of the axis Ox. The rod, which connects the springs, is weightless. The motion of the AB block, which is in the state of rest perpendicular to the axis AB, is consider as translation.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21. At the moment $t = 0$, the load D ($m_D = 2\ kg$) is connected at the points A and B to the free ends of the system of two non-deformed parallel-connected springs have coefficients of stiffness and resistance $c_1 = 7\ N / cm$, $c_2 = 3\ N / cm$, $b_1 = b_2 = 0$. At the same time, the load D give an initial velocity of $v_o = 0.4\ m / s$ in the direction of positive reference X. Take the angle $\alpha = 45^o$.

22. The load D at point A is attached to the spring and held on an inclined plane ($\alpha = 30^o$) in the equilibrium. In this case, the static deformation of the spring is $\delta_{st} = 2\ cm$. At the time $t = 0$, the base at the point B begins to move in accordance with the law $\xi = 0,01\sin 10t$ (m).

23. At the time $t = 0$, the load D ($m_D = 3\ kg$) is connected to the AB beam without initial velocity. Beam AB binds the system of two non-deformed parallel-connected springs, which have coefficients of stiffness and resistance $c_1 = 2\ N / cm$, $c_2 = 4\ N / cm$, $b_1 = b_2 = 6\ Ns / m$. Take the angle $\alpha = 60^o$.

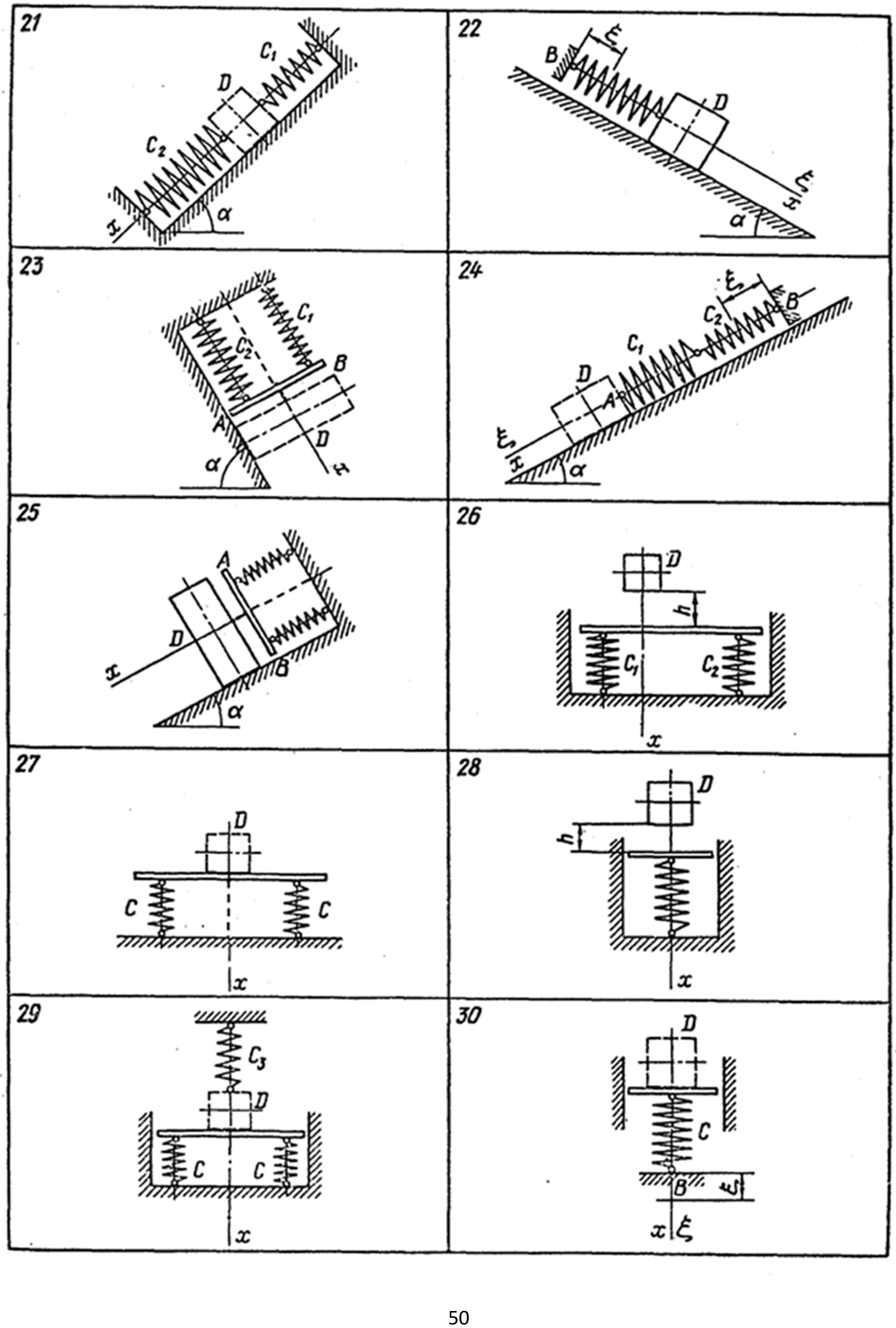
24. At the time $t = 0$, the load D ($m_D = 1\ kg$) is attached at the point A without the initial velocity to the system of two non-deformed series longitudinally connected springs, which have coefficients of stiffness and resistance $c_1 = 12\ N / cm$, $c_2 = 4\ N / cm$, 0. At the same time the base at the point B starts to move along the inclined plane ($\alpha = 30^o$) by law $\xi = 1,5\sin 10t$ (cm).

25. Load D ($m_D = 1,5\ kg$) is attached to a rod AB, suspended on two identical parallel springs. The static deformation of each of the springs with a resistance coefficient $b = 3\ Ns / cm$ under the action of the load D on the inclined plane ($\alpha = 30^o$) is $\delta_{st} =$ 4.9 cm. At the time $t = 0$, the load D, which is in the equilibrium, gives an initial velocity of $v_o = 0.3\ m / s$ in the direction of the negative coordinate X.

Variants 26-30. Find the equation of oscillatory motion of load D in the direction of the axis Ox from the moment of touch to the plate, considering that in further motion the load is not separated from the plate. The plate, which is in equilibrium at the horizontal position, is weightless. The motion of the plate is considered translational.

26. Load D falls down without the initial velocity, distance $h = 0.1\ m$, load D ($m_D = 50$ kg) connects at a time $t = 0$ to a plate which connects a system of two non-deformed parallel-connected springs have coefficients of stiffness and resistance $c_1 = 600\ N / cm$, $c_2 = 400\ N / cm$, $b_1 = b_2 = 0$.

27. At the time $t = 0$, load D ($m_D = 40\ kg$) is set without initial velocity on a slab connecting the system of two identical, in parallel-connected non-deformed springs have each stiffness and resistance coefficients $c = 130\ N / cm$, $b = 200\ Ns / m$.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28. The load D falls on the slab from a height h = 5 cm without initial velocity. The plate fixed on a spring, which static deformation of the applied load is $\delta_{st} = 1$ cm.

29. The plate fixed on two identical parallel springs. At time t = 0, load D (m_D = 200 kg) is set on the plate and suspended to the third spring located above the load. The springs have coefficients of stiffness and resistance $c_1 = c_2 = 400$ N / cm, $c_3 = 200$ N / cm, $b_1 = b_2 = 0$. At the same time, the load gives an initial velocity of $v_o = 0.6$ m / s, in the direction of the positive reference of the coordinate X. At the initial time motion, the three-spring system is in equilibrium.

30. At time t = 0, load D (m_D = 100 kg) is set without initial velocity on a plate fixed to a spring having coefficients of stiffness and resistance b = 0. At the same time, the base at point B begins to move in law $\xi = 0{,}5 \sin 20t$ (cm).

Виробничо-практичне видання

Методичні рекомендації для практичних занять, самостійних
та розрахунково-графічних робіт з навчальної дисципліни

« ТЕОРЕТИЧНА МЕХАНІКА »
«ЧАСТИНА 3 ДИНАМІКА»

(*для студентів-бакалаврів 1 курсу денної форми навчання*
за спеціальністю 192 – Промислове та цивільне будівництво)

(Англійською мовою)

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Відповідальний за випуск *Н. В. Середа*

За авторською редакцією

Комп'ютерне верстання *Є. Г. Панова*

План 2020, поз. 149 М

Підп. до друку 29.05.2020. Формат 60 х 84/16.
Друк на ризографі. Ум. друк. арк. 3,0.
Тираж 50 пр. Зам. №

Видавець і виготовлювач:
Харківський національний університет
міського господарства імені О. М. Бекетова,
вул. Маршала Бажанова, 17, Харків, 61002.
Електронна адреса: rectorat@kname.edu.ua
Свідоцтво суб'єкта видавничої справи:
ДК № 5328 від 11.04.2017.