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VECTOR ALGEBRA

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Посібник містить основні відомості про вектори, дії над векторами, їх властивості, питання для самоконтролю, приклади завдань прикладного характеру, у процесі вирішення яких використовуються розглянуті методи.

Посібник призначений для студентів, викладачів і читачів, які цікавляться питаннями векторної алгебри

Sytnykova Yu. V.

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The tutorial contains basic information about vectors, operations with them, their properties, questions for self-control, and examples of problems of an applied nature in the process of solving which the methods considered are used.

The manual is intended for students, teachers and readers who are interested in questions of vector algebra.

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CONTENT

Overview

May be somebody of you known a good deal already about Vector Algebra as how to add and subtract vectors, how to take scalar and vector products of vectors, and something of how to describe geometric and physical entities using vectors. However, we need to focus your attention on it again, because this topic is important enough for your further vocation technical study.

Vector Algebra studies scalar and vector products; scalar and vector triple products; geometric applications, moreover, how to find the differentiation of a vector function; scalar and vector fields. It is the first step to study what is a gradient, a divergence and curl-definitions and its physical interpretations; product formulae; curvilinear coordinates. How to use Gauss' and Stokes' theorems, and evaluation of integrals over lines, surfaces and volumes. And also you can continue to study more complicated concepts as a derivation of continuity equations and Laplace's equation in Cartesian, cylindrical and spherical coordinate systems and other.

I hope this course will help to revise your knowledge about the concept vector and remind acts with them, combining vector algebra with calculus, and goes on successfully their new application in your special study.

Acknowledgement

The author expresses their gratitude to Svitlana Zubenko, an Associate Professor of the Department of Foreign Languages of O. M. Beketov National University of Urban Economy in Kharkiv for the assistance rendered in the process of preparing this tutorial, for her valuable linguistic remarks and constructive advices in editing the text in English.

I am also grateful to reviewers for their attention to our work, for their comments and their support in the process of working on this tutorial.

THE MAIN CONCEPTS OF VECTOR DEFENITION

Many physical quantities, such as mass, time, temperature, are fully specified by one number or magnitude, because they are scalars. But description of other quantities require more than one number. They are vectors. You have already met vectors in their more pure mathematical sense in your course on linear algebra (matrices and vectors), however, often in the physical world, these numbers specify a magnitude and a direction as a total of two numbers in a 2D planar world, and three numbers in 3D. Obviously, examples of such quantities are velocity, acceleration, electric field, and force. Below, probably all our examples will be of these “magnitude and direction” vectors, but we should not forget that many of the results extend to the wider realm of vectors. There are three slightly different types of vectors:

•*free vectors*: in many situations only the magnitude and direction of a vector are important, and we can translate them at will (with 3 degrees of freedom for a vector in 3-dimensions, because there are three linearly independent planes that the rotation can take place in).

•*sliding vectors*: in mechanics the line of action of a force is often important for deriving moments. The force vector can slide with 1 degree of freedom.

•*bound or position vectors*: when describing lines, curves etc. in space, it is obviously important that the origin and head of the vector will not be transformed about arbitrarily. The origins of position vectors all coincide at an overall origin O.

As we know, a line, which has positive direction, is named *axis*, if there is a starting point at this axis, it is a *coordinate axis*.

Let two points A, B be given on the coordinate axis Ox. If one of these points, for example, A, is taken as the beginning, and the other point, B, as the end of the segment bounded by these points, then this segment is considered as a direction segment.

Definition 1. The *vector* is a directed line segment and it is denoted $\overrightarrow{AB}$ and drawn as arrow (Figure 1).

The vector $\overrightarrow{AB}$ may also be indicated by one Latin letter with an arrow on top, for example: $\vec{a}$ (Figure 1). If we write the vector $\overrightarrow{BA}$, it means that this vector has start at the point B and its end is at the point A. In this case the vector $\overrightarrow{BA}$ (Figure 2) is named *the opposite vector* to the vector $\overrightarrow{AB}$ and

$$\overrightarrow{AB} = -\overrightarrow{BA}.$$

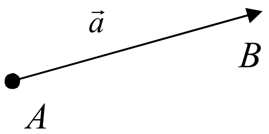

Figure 1

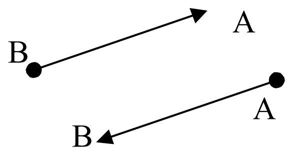

Figure 2

Since the vector is a directed segment, then besides the direction it has, as well as any other segment, *the length*. The length of the vector is understood as such a positive number, which determines the length of the same directed line segment and is denoted by $\left|\overrightarrow{AB}\right|$ or $\left|\vec{a}\right|$. In other words, the length of a vector can be considered as the distance between two points on the coordinate axis Ox, which are its ends. There is fair: $\left|\overrightarrow{AB}\right| = \left|-\overrightarrow{BA}\right|$ (look at the figure 2).

Definition 2. *The opposite vectors* are two vectors with opposite directions and equal lengths.

The number expressing the vector length is also called the vector magnitude, which is a scalar value (or vector modulus).

Definition 3. *Zero vector* is a vector whose beginning and end are the same, $\overrightarrow{AA}$. Denoted by $\vec{0}$ or 0. The zero vector has no direction and his magnitude is zero $|\vec{0}| = 0$. So, we can say that the zero vector is a point.

Definition 4. *A unit vector* (or ort) is named a vector whose length is equal to one.

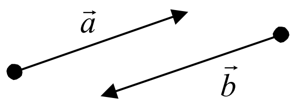

Figure 3

Definition 5. *Collinear vectors* are vectors located on one straight line or on parallel straight lines and denoted by $\vec{a} // \vec{b}$ (Figure 3).

Zero vector is considered collinear to any vector.

If collinear vectors have the same direction, then they are named *unidirectional vectors*, otherwise, they are named *oppositely directed vectors*.

Definition 6. Two vectors are considered *equal* ($\vec{a} = \vec{b}$) if they are equally directed and have the same length.

Note 1. Two vectors are equal only if they can be combined without turning.

LINEAR OPERATIONS OVER VECTORS

Linear operations include adding and subtracting vectors, multiplying the vector by a scalar. Let us consider only free ones vectors.

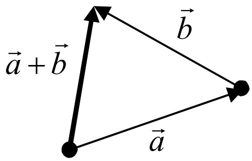

Figure 4

Addition of vectors. We know two rules how to add vectors.

Triangle rule. The sum of vectors $\vec{a}$ and $\vec{b}$ is the third vector $\vec{c}$ ($\vec{c} = \vec{a} + \vec{b}$), the beginning of which coincides with the beginning of the first vector $\vec{a}$, and the end with the end of the second vector $\vec{b}$ (Figure 4).

Similarly, the sum of any finite number of vectors is obtained.

The sum of several vectors is a vector obtained after numerous sequential additions of vectors. That is, such a vector, the beginning of which coincides with the beginning of the first vector-term, and the end of which coincides with the end of the last vector-term (Figure 5)

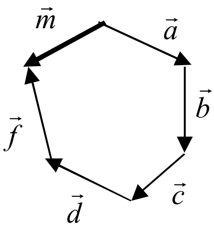

Figure 5

$$\vec{m} = \vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{f}\ .$$

This rule is called the *polygon rule* or the *chain rule*.

Note 2. The sum of the opposite vectors is zero-vector:

$$\vec{a} + \left(-\vec{a}\right) = \vec{0}\,.$$

Rule of parallelogram. If the vectors $\vec{a}$ and $\vec{b}$ are non-collinear vectors, then the sum of the vectors $\vec{a} + \vec{b}$ is determined by the following constructs (Figure 6): we build parallelogram on these vectors. The diagonal of this parallelogram, emerging from the common origin of vectors, will be the sum.

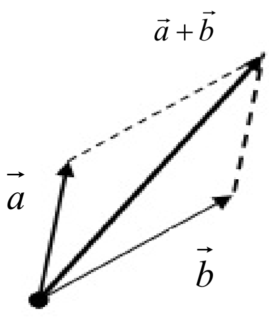

Figure 6

Note 3. The rule of parallelogram is not applicable for collinear vectors.

Subtraction of vectors. As we know the subtraction is an opposite act to the addition, so we can consider it as

$$\vec{a}-\vec{b}=\vec{a}+\left(-\vec{b}\right).$$

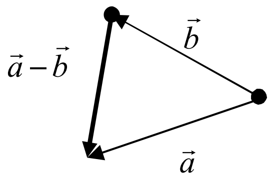

Figure 7

Triangle rule (Figure 7). To get the subtraction of the vectors $\vec{a}$ and $\vec{b}$, it is enough to bring the vectors to the joint start, and then construct a vector whose origin coincides with the beginning of the vector $\vec{b}$, and the end of the vector $\vec{a}-\vec{b}$ is the end of the vector $\vec{a}$.

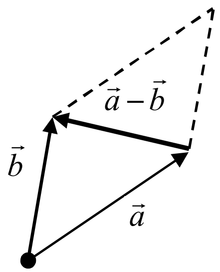

Figure 8

Rule of parallelogram. To get the subtraction of the vectors, it is enough to join them at the start point and construct the parallelogram (Figure 8). Diagonal of the obtained parallelogram is connecting the end point of the vector $\vec{a}$ with the end point of the vector $\vec{b}$ will be the vector of their subtraction $\vec{a}-\vec{b}$.

Note 4. The subtraction magnitude may be less than the minuend magnitude, but may also be greater or equal.

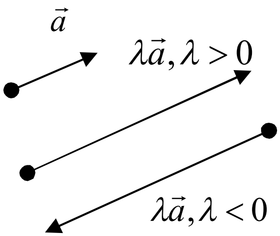

Figure 9

Multiplication of a vector by a scalar. (NOT the scalar product!) To multiply a vector $\vec{a}$ by a scalar λ (which is not equal to zero, $\lambda \neq 0$), it is necessary to construct a new vector (Figure 9), the length of which is multiplied by λ. Its direction coincides with the direction of the vector $\vec{a}$ if

$\lambda > 0$, or has the opposite direction of the vector $\vec{a}$, if $\lambda < 0$.

Properties of linear vector operations:

1. The addition of vectors is commutative:

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}.$$

2. The addition of vectors is associative:

$$\vec{a} + \vec{b} + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right); \ \vec{a} + \vec{b} + \vec{c} = \left(\vec{a} + \vec{b}\right) + \vec{c}.$$

3. The multiplication by a scalar (properties are similar to the basic properties of numbers multiplication):

$$\mu\left(\lambda\vec{a}\right) = (\mu\lambda)\vec{a}; \ \ (\mu + \lambda)\vec{a} = \mu\vec{a} + \lambda\vec{a};$$

$$\mu\left(\vec{a} + \vec{b} - \vec{c}\right) = \mu\vec{a} + \mu\vec{b} - \mu\vec{c}, \ \lambda\vec{a} = 0, \text{ if } \lambda = 0 \text{ or } \vec{a} = 0.$$

4. The sum of any vector and the zero vector:

$$\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}.$$

Note 5. Any vector can be represented as: $\vec{a} = \overrightarrow{a_0} \cdot \left|\vec{a}\right|$, where $\overrightarrow{a_0}$ is an ort, having the same direction with the vector $\vec{a}$. To find the unit vector $\overrightarrow{a_0}$ in the direction of $\vec{a}$, simply divide by its magnitude:

$$\overrightarrow{a_0} = \frac{\vec{a}}{\left|\vec{a}\right|}.$$

Example 1. Simplify the expression

$$\frac{4\vec{a} - 3\vec{b} + 5\vec{c}}{2} + \frac{7\vec{c} - 5\vec{a}}{3}.$$

Solution. Let us lead the given expression to a common denominator

$$\frac{4\vec{a}-3\vec{b}+5\vec{c}}{2}+\frac{7\vec{c}-5\vec{a}}{3}=\frac{12\vec{a}-9\vec{b}+15\vec{c}+14\vec{c}-10\vec{a}}{6}=$$

$$\frac{2\vec{a}-9\vec{b}+29\vec{c}}{6}.$$

Note 6. Actions with vectors are performed similarly to actions with real numbers.

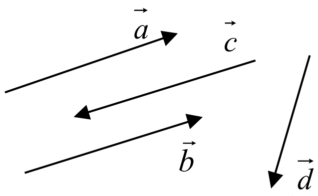

Figure 10

Definition 7. *Collinear vectors* are vectors that lie on one line or are located on parallel lines.

Look at the figure 10, there are some collinear vectors, for example,

$$\vec{a}\,//\,\vec{b},\ \vec{a}\,//\,\vec{c},\ \vec{b}\,//\,\vec{c},$$

but vector $\vec{d}$ is non-collinear to none.

Example 2. In the parallelogram $ABCD$ (Figure 11) are, $\overrightarrow{AB}=\vec{a}$, $\overrightarrow{AD}=\vec{b}$. Express the vectors $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{AC}$, $\overrightarrow{BD}$, $\overrightarrow{DB}$, $\overrightarrow{CA}$ through $\vec{a}$ and $\vec{b}$.

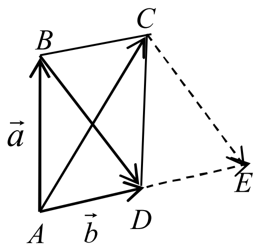

Figure 11

Solution. Vectors $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are collinear vectors having equal lengths (because they are opposite sides of the parallelogram that are equal) and the same direction. According to the definition 6 about the vector equality, we get

$\overrightarrow{BC} = \overrightarrow{AD}$ and $\overrightarrow{AD} = \vec{b} = \overrightarrow{BC}$. Vectors $\overrightarrow{CD}$ and $\overrightarrow{AB}$ are opposite vectors and they have equal lengths. According to the definition 2 about the opposite vectors, we have the following

$$\overrightarrow{CD} = -\overrightarrow{DC} = -\overrightarrow{AB} = -\vec{a}\text{, in this way } \overrightarrow{CD} = -\vec{a}.$$

Vector $\overrightarrow{AC}$ is a sum of vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, but $\overrightarrow{BC} = \overrightarrow{AD}$, so $\overrightarrow{AC} = \vec{a} + \vec{b}$. Further,

$$\overrightarrow{CA} = -\overrightarrow{AC} = -\left(\vec{a} + \vec{b}\right) = -\vec{a} - \vec{b}.$$

According to the definition of two vectors subtraction, we get $\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$ or $\overrightarrow{BD} = \vec{b} - \vec{a}$, because $\overrightarrow{AD} = \vec{b}$, $\overrightarrow{AB} = \vec{a}$.

Similarly, we know that $\overrightarrow{BD}$ and $\overrightarrow{DB}$ are opposite, i.e. $\overrightarrow{DB} = -\overrightarrow{BD}$, but $\overrightarrow{BD} = \vec{b} - \vec{a}$, so we get that

$$\overrightarrow{DB} = -\overrightarrow{BD} = -\left(\vec{b} - \vec{a}\right) = \vec{a} - \vec{b}.$$

Example 3. Prove that any triangle whose sides are equal and parallel to the medians of the given triangle is existed for any triangle.

Solution. Let it be the given triangle ABC (Figure 12). So,

$$\overline{AB} + \overline{BC} + \overline{CA} = 0,$$

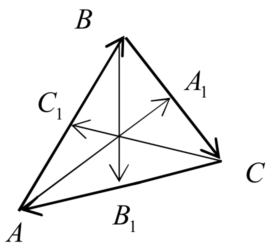

Figure 12

according to the properties of linear vectors operations.

Denote the midpoints of the sides BC, CA, AB by points A_1, B_1, C_1 respectively.

Directed segment can be presented by the following equality

$$\overline{AA_1} = \overline{AB} + \overline{BA_1} = \overline{AB} + \frac{1}{2}\overline{BC}.$$

Similarly,

$$\overline{BB_1} = \overline{BC} + \frac{1}{2}\overline{CA}, \ \overline{CC_1} = \overline{CA} + \frac{1}{2}\overline{AB}.$$

Add all of these equalities and get

$$\overline{AA_1} + \overline{BB_1} + \overline{CC_1} = \overline{AB} + \overline{BC} + \overline{CA} + \frac{1}{2}\left(\overline{AB} + \overline{BC} + \overline{CA}\right) =$$

$$= \frac{3}{2}\left(\overline{AB} + \overline{BC} + \overline{CA}\right) = 0.$$

Thus, $\overline{AA_1} + \overline{BB_1} + \overline{CC_1} = 0$ this is what was required to prove.

Example 4. Prove that the medians of any triangle ABC (Figure 13) intersect at a point M such that: 1) the distance from this point M to each triangle vertices is equal to $\frac{2}{3}$ of the length of the corresponding median; 2) this equality $\overline{OM} = \frac{1}{3}\left(\overline{OA} + \overline{OB} + \overline{OC}\right)$ is truth for any point O

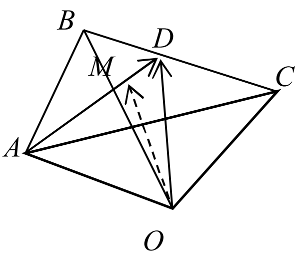

Figure 13

Solution. Let the point M cut off two-thirds of the segment AB from the point A. Then

$$\overline{OM} = \overline{OA} + \overline{AM} = \overline{OA} + \frac{2}{3}\overline{AD} = \overline{OA} + \frac{2}{3}\left(\overline{OD} - \overline{OA}\right) =$$

$$= \overline{OA} + \frac{2}{3}\left(\frac{1}{2}\left(\overline{OB} + \overline{OC}\right) - \overline{OA}\right) = \frac{1}{3}\left(\overline{OA} + \overline{OB} + \overline{OC}\right).$$

We get the same result for any other median of the triangle ABC. It suggests that the point M is a common point for all three medians of the triangle. From the solution of this problem it follows that if the point is intersection point of the medians of any triangle ABC and a point O is an arbitrary point then truth equality makes sense

$$\overline{OM} = \frac{1}{3}\left(\overline{OA} + \overline{OB} + \overline{OC}\right).$$

Definition 8. If the vectors $\vec{a}$ and $\vec{b}$ are collinear vectors, then there is a scalar λ which is named the *ratio of vector* $\vec{b}$ to the collinear vector $\vec{a}$ ($\vec{a} \neq 0$).

In the other words, the *vectors collinearity condition* can be written in a form: $\vec{b} = \lambda\vec{a}$, or $\dfrac{\vec{b}}{\vec{a}} = \lambda$ ($\vec{b} : \vec{a} = \lambda$).

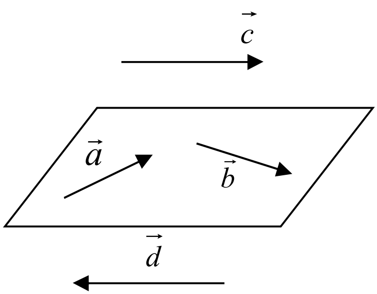

Figure 14

Definition 9. Vectors parallel to the same plane, or lie on the same plane are called *coplanar vectors*.

Coplanar vectors are presented at the figure 14, such vectors are $\vec{a}$ and $\vec{b}$, vectors $\vec{c}$ and $\vec{d}$ are not coplanar vectors, they are not on the same plane.

It is always possible to find a plane parallel to the two random vectors, so any two vectors are always coplanar.

AXIS PROJECTION

The expression "vector projection on the axis Ox" can be used in two senses as geometric and algebraic (arithmetic).

Suppose we have a vector and we need to project this vector on the axis. In order to project a vector, we put down the perpendiculars from the ends of this vector to the intersection with the given axis Ox (Figure 15).

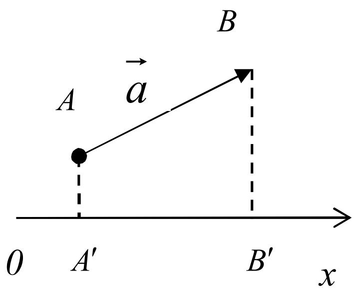

Figure 15

In this case, the vector $\overrightarrow{A'B'}$ is a *geometric projection* of the vector $\overrightarrow{AB}$. It can be denoted

$$proj_{Ox}\overrightarrow{AB} = proj_{Ox}\vec{a} = \overrightarrow{A'B'}.$$

The geometric projection of the vector on the line Ox is also named a *vector component* on the line Ox or *vector coordinate*.

Note 7. Equal vectors have equal components. In case of a parallel vector transfer, its components are also subjected to parallel transfer (along the axis). It is obvious that the components of the same vector in two parallel lines are equal.

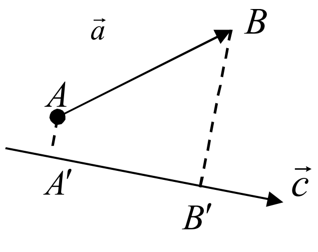

Figure 16

If the axis Ox is presented as a vector $\vec{c}$ (Figure 16), then the vector $\overrightarrow{A'B'}$ is also named the geometric projection of the vector $\overrightarrow{AB}$ on the direction of the vector $\vec{c}$. It is denoted $proj_{\vec{c}}\vec{a}$, $proj_{\vec{c}}\overrightarrow{AB}$.

The *algebraic projection* of the vector $\overrightarrow{AB}$ on the line Ox or vector $\vec{c}$ is algebraic value $A'B'$ of the vector $\overrightarrow{A'B'}$. In the other words, the algebraic projection of the vector is his length taken with a plus or minus sign, depending on whether the projection is aligned with the direction of the axis on which the projection is performed.

Properties of vector projections

1. The projection of the sum of vectors on any axis (or vector) is equal to the sum of the projections of these vectors on the axis (or vector).

For example (Figure 17a),

$$proj_l\left(\overrightarrow{AC} + \overrightarrow{CB}\right) = proj_l \overrightarrow{AC} + proj_l \overrightarrow{CB} .$$

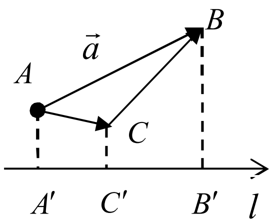

Figure 17a

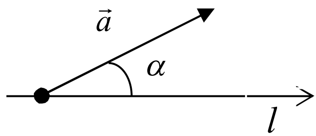

Figure 17b

2. The algebraic projection of a vector on any axis is the product of the length of the vector on the cosine of the angle between the vector and the axis (Figure 17b):

$$proj_l \vec{a} = \left|\vec{a}\right| \cos\alpha .$$

VECTOR ELEMENTS OR COMPONENTS IN A COORDINATE FRAME

As you know, the coordinate method is a way of determining the position of a point or body using numbers or other characters. Depending on the goals and nature of the study, various coordinate frames are chosen. We will use this method to represent the vector as a set of some values. If three pairwise perpendicular lines are drawn through a certain point in space, a direction is selected on each of them (indicated by an arrow) and a unit of measurement for segments is selected, then they say that a rectangular coordinate frame is specified in space. It is the 3-dimensional Cartesian coordinate frame $O(x, y, z)$.

Draw a vector from the origin point O to an arbitrary point $M(x, y, z)$ and project the vector $\overrightarrow{OM} = \vec{a}$ onto coordinate planes (Figure 18). The vector $\overrightarrow{OM}$ like any other vector can be decomposed into three terms that lie on the coordinate axes, so

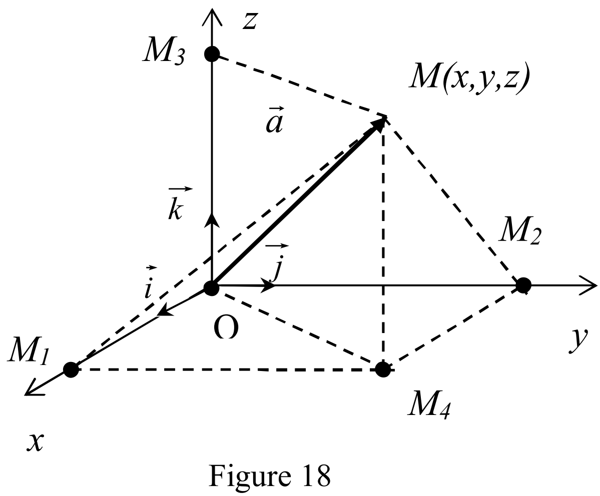

Figure 18

$$\overrightarrow{OM} = \overrightarrow{OM_1} + \overrightarrow{OM_2} + \overrightarrow{OM_3} .$$

Remind you. The Cartesian coordinate frame in space is called the collection of a point and a *basis*. The point is usually denoted by the letter O and is called the *origin*. The straight lines passing through the origin in the direction of the basis vectors are called *axes of coordinates*. Planes that pass through coordinate axes are called *coordinate planes*. A Cartesian coordinate frame is called *rectangular* if all its coordinate axes are perpendicular in pairs.

As we know the vector $\overrightarrow{OM}$ projections $\overrightarrow{OM_1}$, $\overrightarrow{OM_2}$, $\overrightarrow{OM_3}$ onto coordinate axes are $proj_{Ox}\overrightarrow{OM}$, $proj_{Oy}\overrightarrow{OM}$, $proj_{Oy}\overrightarrow{OM}$ accordingly, we also know the points coordinates $M_1(x,0,0)$, $M_2(0,y,0)$, $M_3(0,0,z)$, so we have

$$\overrightarrow{OM_1} = proj_{Ox}\overrightarrow{OM} = x \text{ , } \overrightarrow{OM_2} = proj_{Oy}\overrightarrow{OM} = y \text{ ,}$$

$$\overrightarrow{OM_3} = proj_{Oy}\overrightarrow{OM} = z .$$

If we put unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ (Appendix A) on each of the coordinate axes, then the vector can be written as

$$\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k} ,$$

where x, y, z are coordinates of the vector $\vec{a}$ (write it like this $\vec{a} = (x, y, z)$ and vectors $x\vec{i}$, $y\vec{j}$, $z\vec{k}$ are called *vector elements* or *vector components*. Vector $\overrightarrow{OM}$ (Figure 18), going from origin point O to the given point M is called a *radius-vector* of the point M .

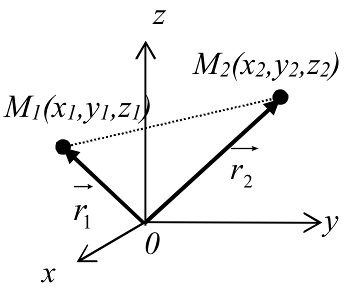

Figure 19

If we have two different points M_1 and M_2 with their radius-vectors $\overrightarrow{OM_1} = \vec{r}_1$, $\overrightarrow{OM_2} = \vec{r}_2$ (Figure 19) respectively then vector $\overrightarrow{M_1M_2} = (x, y, z)$ can be expressed by the formula

$$\overrightarrow{M_1M_2} = \vec{r}_2 - \vec{r}_1 ,$$

it can be rewritten for its coordinates as

$$x = x_2 - x_1 ,\ y = y_2 - y_1 ,\ z = z_2 - z_1 ,$$

that is the vector coordinates are equal to subtraction of the corresponding coordinates its ending and starting point.

The length of the vector (vector magnitude) is calculated by the formula

$$\left|\overrightarrow{M_1M_2}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} .$$

A magnitude of the vector $\vec{a} = (x, y, z)$ (or vector length) can be found

$$\left|\vec{a}\right| = \sqrt{x^2 + y^2 + z^2} .$$

Example 5. Points $M_1(-5,1,-3)$, $M_2(-2,-3,1)$ are given. Find the coordinates and magnitude of the vector $\overrightarrow{M_1M_2}$.

Solution. Coordinates of the desired vector $\overrightarrow{M_1M_2}$ can be found as a subtraction of the corresponded coordinates of the last point and start point, like this

$$\overrightarrow{M_1M_2} = (-2-(-5), -3-1, 1-(-3)) = (3,-4,4).$$

A vector magnitude we find using the formula above

$$\left|\vec{a}\right| = \sqrt{x^2 + y^2 + z^2}, \ \left|\overrightarrow{M_1M_2}\right| = \sqrt{3^2 + (-4)^2 + 4^2} = \sqrt{41}.$$

Note 8. The collinear vectors have proportional coordinates, and vice versa, if the coordinates of two vectors are proportional, then these vectors are parallel (look at definition 8).

Example 6. Vectors $\vec{a} = (2,4,-6)$, $\vec{b} = (-1,-2,3)$, $\vec{c} = (6,0,0)$, $\vec{d} = (0,1,3)$ are given. Which of these vectors are collinear vectors, is parallel to the axis, is parallel to the coordinate plane?

Solution. Since the coordinates of the vectors $\vec{a}$ and $\vec{b}$ are proportional then they are collinear vectors (according to $\vec{b} = \lambda \vec{a}$), they have a proportionality coefficient which is equal to $\frac{-1}{2}$, i.e.

$$\frac{-1}{2} = \frac{-2}{4} = \frac{3}{-6}, \ \vec{b} = \frac{-1}{2}\vec{a}.$$

Comparing the coordinates of the vector $\vec{c}$ with the unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$, we conclude that vector $\vec{c}$ parallel to the x-axis (vector $\vec{c}$ is parallel to the vector $\vec{i}$ located on the x-axis).

Since the vector $\vec{d}$ has a zero x-coordinate ($x = 0$), i.e. its projection on the x-axis is zero, then vector $\vec{d}$ is perpendicular to x-axis and parallel to the coordinate plane $0yz$.

Note 9. If one of the vector coordinates is zero, then this vector is perpendicular to the corresponding coordinate axis. For example, the vector $\vec{a} = (2,1,0)$ is perpendicular to the z-axis, because his applicant is zero ($z = 0$).

Note 10. If a vector has only one nonzero coordinate then this vector is parallel to the corresponding coordinate axis. For example, the vector $\vec{a} = (0,-5,0)$ is parallel to the y-axis.

Example 7. Points M and N are taken on the parallelogram side AD and its diagonal AC, that so $|AM| = \frac{1}{5}|AD|$, $|AN| = \frac{1}{5}|AC|$. Prove that the points M, N, B lie on one line. In what ration does a point N divide a segment MB (figure 20)?

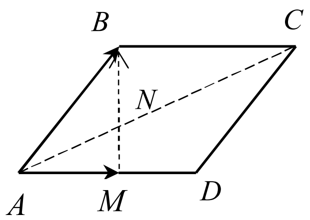

Figure 20

Solution. For proof that the points lie on one line we need to prove that vectors $\overrightarrow{MN}$ and $\overrightarrow{MB}$ are collinear vectors.

We know from the tasks statements that

$$|AM| = \frac{1}{5}|AD|,\ |AN| = \frac{1}{5}|AC|.$$

Then

$$\overrightarrow{MN} = \overrightarrow{AN} - \overrightarrow{AM} = \frac{1}{6}\overrightarrow{AC} - \frac{1}{5}\overrightarrow{AD} =$$

$$= \frac{1}{5}\left(\overrightarrow{AD} + \overrightarrow{DC}\right) - \frac{1}{5}\overrightarrow{AD} = \frac{1}{30}\left(5\overrightarrow{DC} - \overrightarrow{AD}\right).$$

On the other hand,

$$\overrightarrow{MB} = \overrightarrow{AB} - \overrightarrow{AM} = \overrightarrow{AB} - \frac{1}{5}\overrightarrow{AD} = \frac{1}{5}\left(5\overrightarrow{AB} - \overrightarrow{AD}\right).$$

Since $AB = CD$, then it follows from the obtained equality: $\overrightarrow{MB} = 6\overrightarrow{MN}$. It means that the points M, N, B lie on one line, and point N divide a segment MB as $5:1$.

Consider some properties of vectors that are often used in solving geometric tasks.

Let the point C is belonging to the vector $\overrightarrow{AB}$. Obviously, it divides this vector in some ratio $\frac{m}{n}$, i.e., $\frac{\left|\overrightarrow{AC}\right|}{\left|\overrightarrow{CB}\right|}=\frac{m}{n}$, that is possible in case when $\overrightarrow{AC}=\frac{m}{n}\overrightarrow{CB}$ (Figure 21).

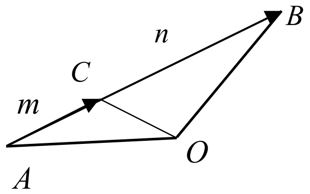

Figure 21

If the points A, B, C are given by their radius-vectors OA, OB, OC relative to some coordinate system. Then from the equality $\overrightarrow{AC}=\frac{m}{n}\overrightarrow{CB}$ we have that

$$\overrightarrow{OC}-\overrightarrow{OA}=\frac{m}{n}\left(\overrightarrow{OB}-\overrightarrow{OC}\right),$$

from this we can find

$$\overrightarrow{OC}=\frac{n}{m+n}\overrightarrow{OA}+\frac{m}{m+n}\overrightarrow{OB}.$$

The written above formula expresses the radius-vector of the point C which divides vector $\overrightarrow{AB}$ in ratio $m:n$ using the radius- vectors of the points A and B.

In particular, if the point C is a midpoint of a vector $\overrightarrow{AB}$, then $\overrightarrow{OC}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OB}\right)$. If the coordinate system 0, $\vec{i}$, $\vec{j}$, $\vec{k}$ is given, and the coordinates of these points are set $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ then the coordinates of point C which divides vector $\overrightarrow{AB}$ in ratio $m:n$, can be expressed by following formulas:

$$x_C = \frac{n}{m+n} x_1 + \frac{m}{m+n} x_2, \ y_C = \frac{n}{m+n} y_1 + \frac{m}{m+n} y_2,$$

$$z_C = \frac{n}{m+n} z_1 + \frac{m}{m+n} z_2 .$$

In the case when the point C divides the vector $\overrightarrow{AB}$ by half, we obtain the following formulas for calculating the coordinates of this point

$$x_C = \frac{x_1 + x_2}{2}, \ y_C = \frac{y_1 + y_2}{2}, \ z_C = \frac{z_1 + z_2}{2} .$$

Note 11. If three points lie on one straight line and are defined with respect to a certain point by their radius-vectors, then the formula is

$$\overrightarrow{OC} = k\overrightarrow{OA} + (1-k)\overrightarrow{OB},$$

where $k = \frac{n}{m+n}$, $1-k = \frac{m}{m+n}$.

However, the converse is also true: the implementation of the above formula guarantees that three points belong to the same straight line. This also follows from the definition of collinearity of vectors.

On the other words, three points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ belong to the one straight line in a space if and only if there is such a number k and it is such that

$$x_3 - x_1 = k(x_2 - x_1), \ y_3 - y_1 = k(y_2 - y_1), \ z_3 - z_1 = k(z_2 - z_1) .$$

In the particular case, when the points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are given on the plane (i.e. in a coordinate system 0, $\vec{i}$, $\vec{j}$), then the previous condition will look like this

$$x_3 - x_1 = k(x_2 - x_1), \ y_3 - y_1 = k(y_2 - y_1) .$$

Since the points $A(x_1, y_1)$ and $B(x_2, y_2)$ do not coincide, either $x_1 \neq x_2$ or $y_1 \neq y_2$. Let's suppose that $x_1 \neq x_2$, then find k:

$$k = \frac{x_3 - x_1}{x_2 - x_1} \text{ and } y_3 - y_1 = \frac{x_3 - x_1}{x_2 - x_1}(y_2 - y_1).$$

Consequently,

$$(y_3 - y_1)\cdot(x_2 - x_1) = (x_3 - x_1)\cdot(y_2 - y_1).$$

We get this result when we suppose that $y_1 \neq y_2$. Thus, this is the condition under which three points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ lie on one straight line. Obviously, if $x_1 \neq x_2$ and $y_1 \neq y_2$ that condition is equivalent to the following condition

$$\frac{x_3 - x_1}{x_2 - x_1} = \frac{y_3 - y_1}{y_2 - y_1}.$$

Example 8. Determine the coordinates of the vector end-point B if the coordinates of the beginning point $A(-2,1,5)$ and midpoint $C(-3,6,1)$ of this vector are known.

Solution. Use these formulas

$$x_C = \frac{x_1 + x_2}{2},\ y_C = \frac{y_1 + y_2}{2},\ z_C = \frac{z_1 + z_2}{2},$$

where $x_C = -3$, $y_C = 6$, $z_C = 1$, $x_1 = -2$, $y_1 = 1$, $z_1 = 5$. Express unknown coordinates of a point $B(x_2, y_2, z_2)$ and get the following formulas:

$$x_C = \frac{x_1 + x_2}{2},\ 2x_C = x_1 + x_2,\ x_2 = 2x_C - x_1;$$

$$y_C = \frac{y_1 + y_2}{2},\ 2y_C = y_1 + y_2,\ y_2 = 2y_C - y_1;$$

$$z_C = \frac{z_1 + z_2}{2},\ 2z_C = z_1 + z_2,\ z_2 = 2z_C - z_1.$$

Substitute the given values of the coordinates of the points and calculate: $x_2 = 2x_C - x_1 = -6 - 2 = -8$. Similarly, $y_2 = 2y_C - y_1 = 12 - 1 = 11$, $z_2 = 2 - 5 = -3$. So, coordinates of the end-point B is $(-8,11,-3)$.

Example 9. The straight line MN is given by the coordinates of the points $M(5,0,1)$ and $N(4,1,-2)$. At what values x and y will belong the point $K(x, y,4)$ to the line MN?

Solution. Use formulas of calculation of the coordinates of the point belonging to the line (note 11), namely,

$$x_3 - x_1 = k(x_2 - x_1),\ y_3 - y_1 = k(y_2 - y_1),\ z_3 - z_1 = k(z_2 - z_1).$$

In our case we have

$$x - 5 = k(4 - 5),\ y - 0 = k(1 - 0),\ 4 - 1 = k(-2 - 1),$$

and get

$$k = -1,\ x = 6,\ y = -1.$$

So, the point K will belong to the straight line MN, $K \in MN$, if $x = 6$, $y = -1$.

ACTIONS ON VECTORS GIVEN BY THEIR COORDINATES

Consider linear operations with vectors for the case when the vectors are given in coordinate form.

Definition 10. The coordinates of the vectors sum are equal to the sums of the corresponding coordinates of the vectors terms.

For example, vectors $\vec{a} = (x_1, y_1, z_1)$ and $\vec{b} = (x_2, y_2, z_2)$ are given, and we need to find the vector of their sum

$$\vec{a} + \vec{b} = (x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

Definition 11. The coordinates of the vectors subtraction are equal to the subtractions of the corresponding coordinates of the vectors terms.

For example, vectors $\vec{a} = (x_1, y_1, z_1)$ and $\vec{b} = (x_2, y_2, z_2)$ are given, and we need to find the vector of their subtraction. It will be

$$\vec{a} - \vec{b} = (x_1 - x_2, y_1 - y_2, z_1 - z_2).$$

Example 10. Find the coordinates of the vector $\vec{a}$, if $\vec{a} = 2\vec{c} + 3\vec{d}$, and $\vec{c} = (1,-2,-4)$, $\vec{d} = (-5,0,8)$.

Solution. Vector $\vec{a}$ is a sum of vectors $2\vec{c}$ and $3\vec{d}$. Find them using definition 10 and 12.

$$2\vec{c} = 2 \cdot (1,-2,-4) = (2,-4,-8);\ 3\vec{d} = (-3 \cdot 5, 0, 3 \cdot 8) = (-15,0,24);$$

$$\vec{a} = 2\vec{c} + 3\vec{d} = (2,-4,-8) + (-15,0,24) = (-13,-4,16).$$

Example 11. Find the magnitude of the vector $\vec{q}$, if $\vec{q} = \overrightarrow{AB} - 3\overrightarrow{CD}$, $A(1,-2,-4)$, $B(1,0,-2)$, $C(5,-1,1)$, $D(0,-2,6)$.

Solution. Vector $\vec{q}$ is a subtraction of vectors $\overrightarrow{AB}$ and $3\overrightarrow{CD}$. Find them (look at examples 2 and 3)

$$\overrightarrow{AB} = (1-1, 0+2, -2+4) = (0,2,2);$$
$$\overrightarrow{CD} = (0-5, -2+1, 6-1) = (-5,-1,5),$$
$$3\overrightarrow{CD} = (-5\cdot 3, -1\cdot 3, 5\cdot 3) = (-15,-3,15);$$
$$\vec{q} = \overrightarrow{AB} - 3\overrightarrow{CD} = (0,2,2) - (-15,-3,15) = (15,5,-13);$$
$$|\vec{q}| = \sqrt{15^2 + 5^2 + (-13)^2} = \sqrt{225+25+169} = \sqrt{419}.$$

Definition 12. The coordinates of the vector $\vec{a}$ multiplication by the scalar λ are equal to the multiplication of the coordinates of the vector $\vec{a}$ by this scalar λ.

For example, vector $\vec{a} = (x_1, y_1, z_1)$ is given, and we need to find the vector $\vec{b} = \vec{a}\lambda$. Accordingly to the mention above definition, we have to multiply each coordinate of the vector $\vec{a}$ by this scalar λ, so

$$\vec{b} = \vec{a}\lambda = (\lambda x_1, \lambda y_1, \lambda z_1).$$

Note 12. Moreover, these found vectors $\vec{a}$ and $\vec{b}$ will be collinear (based on the collinearity condition above).

Example 12. Check, are vectors $\vec{c_1}$ and $\vec{c_2}$ built on vectors $\vec{a} = (1,-2,2)$ and $\vec{b} = (3,0,-1)$ collinear, if $\vec{c_1} = 2\vec{a} + 4\vec{b}$, $\vec{c_2} = 3\vec{b} - \vec{a}$.

Solution. Define the coordinates of the vectors $\vec{c_1}$ and $\vec{c_2}$, which are linear combinations of vectors $\vec{a}$ and $\vec{b}$:

$$2\vec{a} = 2\cdot\overline{(1,-2,2)} = (2,-4,4),\ 4\vec{b} = 4\cdot\overline{(3,0,-1)} = (12,0,-4),$$

$$\overrightarrow{c_1} = 2\vec{a} + 4\vec{b} = (2 + 12, -4 + 0, 4 - 4) = (14, -4, 0);$$

$$3\vec{b} = 3 \cdot \overline{(3,0,-1)} = (9,0,-3),\ \vec{a} = (1,-2,2),$$

$$\overrightarrow{c_2} = 3\vec{b} - \vec{a} = (9 - 1, 0 - (-2), -3 - 2) = (8,2,-5).$$

Use the collinearity condition and note 12, we have to get the correct equalities

$$\frac{\overrightarrow{c_1}}{\overrightarrow{c_2}} = \lambda \text{ or } \frac{x_{\vec{c_1}}}{x_{\vec{c_2}}} = \frac{y_{\vec{c_1}}}{y_{\vec{c_2}}} = \frac{z_{\vec{c_1}}}{z_{\vec{c_2}}} = \lambda .$$

Compare the obtained coordinates of vectors $\vec{c_1}$ and $\vec{c_2}$, and get

$$\frac{14}{8} \neq \frac{-4}{2} \neq \frac{0}{-5} \neq \lambda ,$$

from here we can make conclusion that the vectors $\vec{c_1}$ and $\vec{c_2}$ are not collinear.

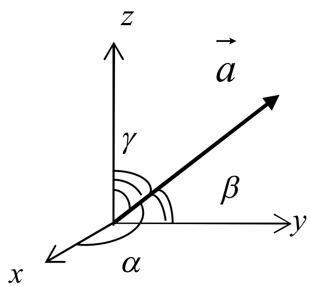

Figure 22

Let's go on to consider the angles formed by a vector with coordinate axes. Look at the figure 22. We can see three angels which were formed by the vector $\vec{a}$ with the x-axis (angel α), y-axis (angel β), and z-axis (angel γ). Angels α, β and γ are called *direction angels* and the cosines of these angels are called *direction cosines* for which the following equality is correct:

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

If the vector $\vec{a}$ is given with its coordinates $\vec{a}=(x,y,z)$, the formulas for the search of the direction cosines are,

$$\cos\alpha=\frac{x}{\left|\vec{a}\right|},\ \cos\beta=\frac{y}{\left|\vec{a}\right|},\ \cos\gamma=\frac{z}{\left|\vec{a}\right|};\text{where } \left|\vec{a}\right|=\sqrt{x^2+y^2+z^2}.$$

Note 13. Coordinates of the vector $\vec{a}=(x,y,z)$ expressed in terms of the direction cosines are:

$$x=\left|\vec{a}\right|\cos\alpha,\ y=\left|\vec{a}\right|\cos\alpha,\ z=\left|\vec{a}\right|\cos\alpha;$$

coordinates of the unit vector $\vec{e}$ equal to the direction cosines

$$\vec{e}=(\cos\alpha,\cos\beta,\cos\gamma).$$

Example 13. Vector $\vec{a}$ forms with the y-axis and z-axis angles equaled to 60^0. Find the angle between the vector $\vec{a}$ and the x-axis.

Solution. As we know $\beta=\gamma=60^0$, then $\cos\beta=\cos\gamma=\cos 60^0=\frac{1}{2}$. Use the equality for direction angles: $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$. Find the direction cosines for angel β and angel γ:

$$\cos^2\alpha+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2=1 \text{ or } \cos^2\alpha=\frac{1}{2},$$

from here

$$\cos\alpha=\pm\frac{1}{\sqrt{2}},\ \cos\alpha_1=\frac{1}{\sqrt{2}},\ \cos\alpha_2=-\frac{1}{\sqrt{2}}.$$

Therefore, we have two answers: $\alpha_1=45^0$, $\alpha_2=135^0$.

SCALAR PRODUCT OF VECTORS

Definition 13. The scalar product of two vectors is a scalar equals to the product of magnitudes of these vectors and the cosine of the angle between them (Figure 23).

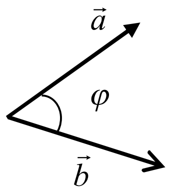

Figure 23

It is denoted $\vec{a}\cdot\vec{b}$ or $\left(\vec{a},\vec{b}\right)$ and calculated by the formula:

$$\vec{a}\cdot\vec{b}=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\cos\varphi,$$

where φ is angel between two vectors $\vec{a}$ and $\vec{b}$.

Note 13. The sign of a scalar product depends on the angle formed by the vectors $\vec{a}$ and $\vec{b}$. If the angle φ is an obtuse angle (in this case, $90^0<\varphi<180^0$ and $\cos\varphi<0$) then the scalar product has a negative sign, otherwise, if the angle φ is an acute angle ($0^0<\varphi<90^0$, $\cos\varphi>0$) then the scalar product has a positive sign.

Geometrical interpretation of a scalar product

Analyzing the graphical representation of cases when the angle between the vectors is acute (Figure 24a) and obtuse (Figure 24b), we can describe the scalar product as the product

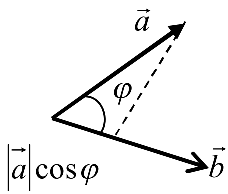

Figure 24a

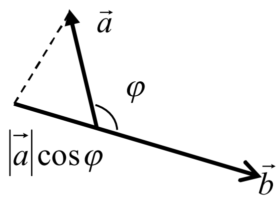

Figure 24b

of the magnitude of one vector and the component of the other in the direction of the first one, since $\left|\vec{a}\right|\cos\varphi$ is the component of $\vec{a}$ in the direction of $\vec{b}$ and also $\left|\vec{b}\right|\cos\varphi$ is the component of $\vec{b}$ in the direction of $\vec{a}$. So, the scalar product can be rewritten in the projection form

$$\vec{a}\cdot\vec{b}=\left|\vec{b}\right|\cdot proj_{\vec{b}}\vec{a}=\left|\vec{a}\right|\cdot proj_{\vec{a}}\vec{b}.$$

Consider the case when the angle between the vectors is right angle, i.e. $\varphi=90^0$, then $\cos\varphi=\cos 90^0=0$, and the scalar product equals zero. So, we came to the following conclusion that if vectors are perpendicular then their scalar product is zero, and vice versa,

$$\varphi=90^0,\ \vec{a}\perp\vec{b}\ \Leftrightarrow\ \vec{a}\cdot\vec{b}=0.$$

It is called the *orthogonality (perpendicularity) condition*.

Properties of a scalar product

1. The scalar product is commutative: it does not change after transposition places of terms,

$$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}.$$

2. The scalar product is distributive over vector addition,

$$\vec{a}\cdot\left(\vec{b}+\vec{c}\right)=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}.$$

3. The scalar product of two vectors multiple by a scalar,

$$\lambda\left(\vec{a}\cdot\vec{b}\right)=\left(\lambda\vec{a}\right)\cdot\vec{b}=\vec{a}\cdot\left(\lambda\vec{b}\right).$$

Note 15. The aforementioned properties make it possible to apply the same transformations, which were applied in algebra to ordinary pair products of factors of the first degree, to the scalar product.

Example 14. Simplify the expression $\left(2\vec{a}-3\vec{b}\right)\left(\vec{c}+5\vec{d}\right)$.

Solution. Open the parentheses, performing multiplication

$$\left(2\vec{a}-3\vec{b}\right)\left(\vec{c}+5\vec{d}\right)=2\vec{a}\vec{c}-3\vec{b}\vec{c}+2\vec{a}5\vec{d}-3\vec{b}5\vec{d}=$$
$$=2\vec{a}\vec{c}-3\vec{b}\vec{c}+10\vec{a}\vec{d}-15\vec{b}\vec{d}\ .$$

Example 15. Factor the expression $6\vec{a}\vec{b}+9\vec{a}\vec{c}$.

Solution. As we see both terms of the expression have a common factor $3\vec{a}$, put it outside the parentheses

$$6\vec{a}\vec{b}+9\vec{a}\vec{c}=3\vec{a}\left(2\vec{b}+3\vec{c}\right).$$

4. If vectors $\vec{a}$ and $\vec{b}$ are collinear vectors then their scalar product equals product its magnitudes taken with a plus or minus sign, depending on whether they are directed in the same direction or opposite one:

$$\vec{a}\cdot\vec{b}=\pm\left|\vec{a}\right|\cdot\left|\vec{b}\right|.$$

Really, if vectors $\vec{a}$ and $\vec{b}$ have the same direction, i.e. $\varphi=0^0$ and $\cos\varphi=1$, then $\vec{a}\cdot\vec{b}=\left|\vec{a}\right|\cdot\left|\vec{b}\right|$; if vectors $\vec{a}$ and $\vec{b}$ have the opposite direction i.e. $\varphi=180^0$ and $\cos\varphi=-1$, then $\vec{a}\cdot\vec{b}=-\left|\vec{a}\right|\cdot\left|\vec{b}\right|$.

5. The scalar square of a vector is equal to the square of its magnitudes:

$$\vec{a}^{\,2}=\vec{a}\cdot\vec{a}=\left|\vec{a}\right|\cdot\left|\vec{a}\right|=\left|\vec{a}\right|^2.$$

Note 16. According to the property 5, the *scalar square* $\vec{a}^{\,2}$ of the vector $\vec{a}$ is a positive number (or zero). So, we can find roots of any power from this number. In particular, we can calculate the square root,

$$\sqrt{\vec{a}^{\,2}}=\left|\vec{a}\right|,$$

that is to say, the arithmetic square root of the scalar product is the vector magnitude.

Note 17. There is no scalar cube in vector algebra, since the scalar product cannot be applied to three factors.

Note 18. The scalar product of unit vectors:

$$\vec{i}\cdot\vec{i}=\vec{i}^{\,2}=\left|\vec{i}\right|^2=1,\ \vec{j}^{\,2}=1,\ \vec{k}^{\,2}=1;$$

$$\vec{i}\cdot\vec{j}=0\ \left(\vec{j}\perp\vec{i}\right),\ \vec{i}\cdot\vec{k}=0\ \left(\vec{k}\perp\vec{i}\right),\ \vec{j}\cdot\vec{k}=0\ \left(\vec{j}\perp\vec{k}\right).$$

Let's consider how to find the scalar product of two vectors when these vectors are given in Cartesian form, for instance, $\vec{a}=\left(x_1,y_1,z_1\right)$ and $\vec{b}=\left(x_2,y_2,z_2\right)$. The scalar product is expressed by formula

$$\vec{a}\cdot\vec{b}=x_1\cdot x_2+y_1\cdot y_2+z_1\cdot z_2\,.$$

Find the angel between vectors $\vec{a}$ and $\vec{b}$. Since the scalar product can be calculated by using formula from the definition 13,

$$\vec{a}\cdot\vec{b}=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\cos\varphi\,,$$

express the cosine of angel φ

$$\cos\varphi=\frac{\vec{a}\cdot\vec{b}}{\left|\vec{a}\right|\cdot\left|\vec{b}\right|}\,,$$

i.e. the cosine of angel between two vectors equals ratio of the scalar product of these vectors to the product of their magnitudes.

If vectors are given in Cartesian form, we can use this formula

$$\cos\varphi=\frac{x_1\cdot x_2+y_1\cdot y_2+z_1\cdot z_2}{\sqrt{x_1^{\,2}+y_1^{\,2}+z_1^{\,2}}\cdot\sqrt{x_2^{\,2}+y_2^{\,2}+z_2^{\,2}}}\,.$$

Example 16. Prove that the vectors $\vec{p}=\vec{a}-\dfrac{\vec{b}\left(\vec{b}\vec{a}\right)}{\vec{b}^{\,2}}$ and $\vec{b}$ are orthogonal.

Solution. As we know, vectors are orthogonal if their scalar product is equal to zero. Check this, multiply the vectors

$$\vec{b}\vec{p}=\vec{b}\left(\vec{a}-\frac{\vec{b}\left(\vec{b}\vec{a}\right)}{\vec{b}^2}\right)=\vec{b}\vec{a}-\frac{\vec{b}\vec{b}\left(\vec{b}\vec{a}\right)}{\vec{b}^2}=\vec{b}\vec{a}-\frac{\vec{b}^2\left(\vec{b}\vec{a}\right)}{\vec{b}^2}=\vec{b}\vec{a}-\vec{b}\vec{a}=0\,.$$

Since $\vec{b}\vec{p}=0$ then $\vec{b}$ and $\vec{p}$ are orthogonal vectors ($\vec{b}\perp\vec{p}$).

Example 17. Find the projection of the vector $\overrightarrow{AB}$ on the direction of the vector $\overrightarrow{CD}$ if points $A(-2,1,3)$, $B(2,-1,7)$, $C(1,2,-5)$, $D(0,5,3)$ are given.

Solution. Use the geometrical interpretation of a scalar product, namely,

$$\vec{a}\cdot\vec{b}=\left|\vec{b}\right|\cdot proj_{\vec{b}}\vec{a}=\left|\vec{a}\right|\cdot proj_{\vec{a}}\vec{b}\,,$$

for our task it looks like

$$proj_{\overrightarrow{CD}}\overrightarrow{AB}=\frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{\left|\overrightarrow{CD}\right|}\,.$$

Find coordinates of vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ (look at the example 2), their scalar product and magnitude of the vector $\overrightarrow{CD}$:

$$\overrightarrow{AB}=(2+2,-1-1,7-3)=(4,-2,4);\ \overrightarrow{CD}=(-1,3,8);$$

$$\overrightarrow{AB}\cdot\overrightarrow{CD}=4\cdot(-1)+(-2)\cdot 3+4\cdot 8=-4-6+32=22\,;$$

$$\left|\overrightarrow{CD}\right|=\sqrt{(-1)^2+3^2+8^2}=\sqrt{1+9+64}=\sqrt{74}\,.$$

Find the projection of the vector $\overrightarrow{AB}$ on the direction of the vector $\overrightarrow{CD}$:

$$proj_{\overrightarrow{CD}}\overrightarrow{AB} = \frac{22}{\sqrt{74}}.$$

Note that the obtained value of projection has a positive sign, it means that the projection has the same direction with the vector on which it is projected, and also, analyzing the scalar product sign, you can say that the angel between the given vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is an acute angel.

Example 18. Find the angel between vectors $\vec{a} = 2\vec{m} + 4\vec{n}$ and $\vec{b} = \vec{m} - \vec{n}$, where $|\vec{m}| = |\vec{n}| = 1$, and angel $\angle(\vec{m},\vec{n}) = 120^0$.

Solution. Calculate the angel between two vectors (remember that $|\vec{m}|^2 = |\vec{n}|^2 = 1$) by the formula

$$\cos\varphi = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|},\ |\vec{a}| = \sqrt{\vec{a}^2} = \sqrt{\left(2\vec{m} + 4\vec{n}\right)^2} =$$

$$= \sqrt{4\vec{m}^2 + 16\vec{m}\vec{n} + 16\vec{n}^2} = \sqrt{4 + 16\vec{m}\vec{n} + 16} = \sqrt{20 + 16\vec{m}\vec{n}};$$

$$|\vec{b}| = \sqrt{\vec{b}^2} = \sqrt{\left(\vec{m} - \vec{n}\right)^2} = \sqrt{\vec{m}^2 - 2\vec{m}\vec{n} + \vec{n}^2} = \sqrt{2 - 2\vec{m}\vec{n}};$$

$$\vec{a}\cdot\vec{b} = \left(2\vec{m} + 4\vec{n}\right)\cdot\left(\vec{m} - \vec{n}\right) = 2\vec{m}^2 + 4\vec{n}\vec{m} - 2\vec{m}\vec{n} - 4\vec{n}^2 = 2\vec{m}\vec{n} - 2;$$

$$\vec{m}\cdot\vec{n} = |\vec{m}|\cdot|\vec{n}|\cdot\cos 120^0 = -\frac{1}{2};\ |\vec{b}| = \sqrt{2 - 2\vec{m}\vec{n}} = \sqrt{3};$$

$$|\vec{a}| = \sqrt{20 + 16\vec{m}\vec{n}} = \sqrt{20 - 8} = \sqrt{12} = 2\sqrt{3};\ \vec{a}\cdot\vec{b} = 2\vec{m}\vec{n} - 2 = -3.$$

Finally, we get the answer: $\cos\varphi = \frac{-3}{2\sqrt{3}\cdot\sqrt{3}} = -\frac{1}{2}$, then

$$\varphi = \arccos\left(-\frac{1}{2}\right) = \pi - \arccos\frac{1}{2} = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \text{ or } \varphi = 120^0.$$

Example 19. Prove that the sum of the parallelogram diagonals squares is equal to the sum of its sides squares.

Solution. Let vectors $\vec{a}$, $\vec{b}$, $\overrightarrow{d_1}$ and $\overrightarrow{d_2}$ be parallelogram sides and diagonals respectively, and $\overrightarrow{d_1} = \vec{a} + \vec{b}$, and $\overrightarrow{d_2} = \vec{a} - \vec{b}$. Draw vectors $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$ from the point O (Figure 25). Then, obviously, $\overrightarrow{d_1} = \overrightarrow{OC} = \vec{a} + \vec{b}$ and $\overrightarrow{d_2} = \overrightarrow{BA} = \vec{a} - \vec{b}$. We have the following for scalar squares of diagonals $\overrightarrow{d_1}$ and $\overrightarrow{d_2}$:

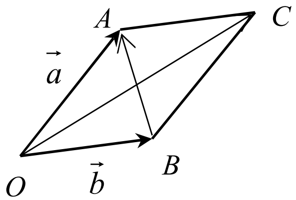

Figure 25

$$\overrightarrow{d_1}^2 = \left(\vec{a} + \vec{b}\right)^2 = \vec{a}^2 + 2\vec{a}\vec{b} + \vec{b}^2;$$

$$\overrightarrow{d_2}^2 = \left(\vec{a} - \vec{b}\right)^2 = \vec{a}^2 - 2\vec{a}\vec{b} + \vec{b}^2.$$

Add these equalities and get:

$$\overrightarrow{d_1}^2 + \overrightarrow{d_2}^2 = 2\left(\vec{a}^2 + \vec{b}^2\right).$$

Whereas,

$$\overrightarrow{d_1}^2 = \left|\overrightarrow{d_1}\right|^2,\ \overrightarrow{d_2}^2 = \left|\overrightarrow{d_2}\right|^2,\ \vec{a}^2 = \left|\vec{a}\right|^2,\ \vec{b}^2 = \left|\vec{b}\right|^2,$$

so

$$\left|\overrightarrow{d_1}\right|^2 + \left|\overrightarrow{d_1}\right|^2 = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2,$$

as required to prove.

Example 20. Prove that the heights in an arbitrary triangle intersect at one point.

Solution. Let ABC be an arbitrary triangle, it has the heights AD and BE, and $AD \perp BC$, $BE \perp AC$, and also a point O is the intersection point of the triangle heights (Figure 26).

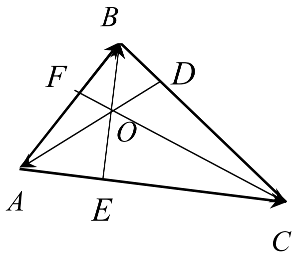

Figure 26

Introduce some denotation: $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$, and F is the intersection point of AB and CO.

According to the definition of the vectors subtraction we get

$$\overrightarrow{AB} = \vec{b} - \vec{a},\ \overrightarrow{BC} = \vec{c} - \vec{b},$$
$$\overrightarrow{CA} = \vec{a} - \vec{c}.$$

In that $\overrightarrow{AD} \perp \overrightarrow{BC}$ according to the orthogonality condition we have $\vec{a} \cdot \left(\vec{c} - \vec{b}\right) = 0$ this implies: $\vec{a} \cdot \vec{c} = \vec{a} \cdot \vec{b}$. Similarly, $\overrightarrow{OB} \perp \overrightarrow{AC}$, so

$$\vec{b} \cdot \left(\vec{a} - \vec{c}\right) = 0 \Rightarrow \vec{b} \cdot \vec{c} = \vec{a} \cdot \vec{b}.$$

Comparing the obtained equalities for scalar products of vectors, we get

$$\vec{a} \cdot \vec{c} = \vec{c} \cdot \vec{b} \Rightarrow \vec{c} \cdot \left(\vec{a} - \vec{b}\right) = 0.$$

Since $\overrightarrow{OC} = \vec{c}$, $\overrightarrow{AB} = \vec{b} - \vec{a}$ then taking into account the previous equality we get that $\overrightarrow{OC} \perp \overrightarrow{AB}$ thus CF is a height of this triangle. Hence, the heights in an arbitrary triangle intersect at one point.

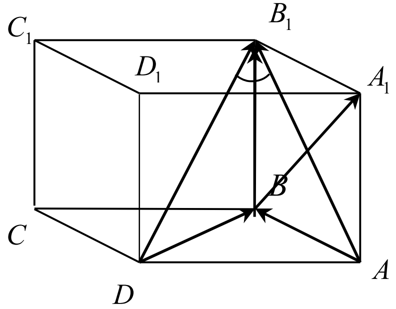

Figure 27

Example 21. Find value of the angles between the cube diagonal and its side faces diagonals (Figure 27).

Solution. Let $\overrightarrow{DB_1}$ be a cube diagonal and $\overrightarrow{AB_1}$, $\overrightarrow{BA_1}$ are its side faces diagonals (Figure 27). Set up the coordinate system A, $\vec{i}$, $\vec{j}$, $\vec{k}$ so that $\overrightarrow{AD} = a\vec{i}$, $\overrightarrow{AB} = a\vec{j}$, $\overrightarrow{AA_1} = a\vec{k}$ where a is a cube edge length.

Express vectors $\overrightarrow{AB_1}$, $\overrightarrow{BA_1}$ and $\overrightarrow{DB_1}$ through vectors $\vec{i}$, $\vec{j}$, $\vec{k}$:

$$\overrightarrow{AB_1} = \overrightarrow{AB} + \overrightarrow{AA_1} = a\vec{j} + a\vec{k} = 0\vec{i} + a\vec{j} + a\vec{k};$$

$$\overrightarrow{BA_1} = -\overrightarrow{AB} + \overrightarrow{AA_1} = -a\vec{j} + a\vec{k} = 0\vec{i} - a\vec{j} + a\vec{k};$$

$$\overrightarrow{DB_1} = \overrightarrow{DB} + \overrightarrow{AA_1} = \overrightarrow{AB} - \overrightarrow{AD} + \overrightarrow{AA_1} = a\vec{j} - a\vec{i} + a\vec{k}.$$

Thus, the vectors $\overrightarrow{AB_1}$, $\overrightarrow{BA_1}$ and $\overrightarrow{DB_1}$ in defined by us coordinate system has such coordinates

$$\overrightarrow{AB_1} = (0, a, a);\ \overrightarrow{BA_1} = (0, -a, a);\ \overrightarrow{DB_1} = (-a, a, a).$$

Use the formula to calculate the cosine of the angle between the vectors and get

$$\cos\varphi = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|},\ \cos\left(\overrightarrow{AB_1}, \overrightarrow{DB_1}\right) = \frac{\overrightarrow{AB_1}\cdot\overrightarrow{DB_1}}{\left|\overrightarrow{AB_1}\right|\cdot\left|\overrightarrow{DB_1}\right|} =$$

$$= \frac{0 + a^2 + a^2}{\sqrt{0 + a^2 + a^2}\cdot\sqrt{(-a)^2 + a^2 + a^2}} = \frac{2a^2}{\sqrt{2a^2}\cdot\sqrt{3a^2}} = \frac{2a^2}{a^2\sqrt{6}} = \frac{\sqrt{6}}{3};$$

knowing the cosine value, we can find the angle value using the corresponding tables. Similarly, we find the angle value between the vectors

$$\cos\left(\overrightarrow{BA_1}, \overrightarrow{DB_1}\right) = \frac{\overrightarrow{BA_1}\cdot\overrightarrow{DB_1}}{\left|\overrightarrow{BA_1}\right|\cdot\left|\overrightarrow{DB_1}\right|} = \frac{0 + (-a)a + a^2}{\sqrt{(-a)^2 + a^2}\cdot\sqrt{(-a)^2 + a^2 + a^2}} = 0,$$

from here we can conclude that

$$\cos\left(\overrightarrow{BA_1}, \overrightarrow{DB_1}\right) = 0 \Rightarrow \angle\left(\overrightarrow{BA_1}, \overrightarrow{DB_1}\right) = 90^0.$$

Physical interpretation of a scalar product

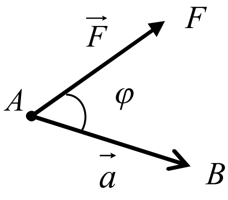

Figure 28

Consider the vector $\vec{a} = \overrightarrow{AB}$ which is drawing rectilinear displacement of a material point and the vector $\vec{F} = \overrightarrow{AF}$ is a vector of a force (Figure 28) acting at this point, then the scalar product $\vec{F} \cdot \vec{a}$ is equal numerically to work A of a force $\vec{F}$. However, during calculating a work A, only force component on the vector $\vec{a}$ and also the magnitude $|\vec{a}|$ are taken into account, i.e. $proj_{\vec{a}}\vec{F}$, so $A = |\vec{a}| proj_{\vec{a}}\vec{F}$ or $A = \vec{a} \cdot \vec{F}$.

Example 22. A material point weighing $20g$ has shifted due to the gravity along a plane inclined at the horizon at an angle 45^0 and located at a distance $0{,}04m$.

Solution. A weight force ($20g = 0{,}02kg$) is drawing by the vector $\overrightarrow{AF}$ (Figure 28) with a length $0{,}02m$. The displacement of the point is represented by the vector $\overrightarrow{AB}$ with a length $0{,}04$. The angel between vectors $\overrightarrow{AF}$ and $\overrightarrow{AB}$ is 45^0:

$$|\vec{F}| = |\overrightarrow{AF}| = 0{,}02; \ |\vec{a}| = |\overrightarrow{AB}| = 0{,}04; \ \angle\left(\vec{a}, \vec{F}\right) = 45^0.$$

Calculate the scalar product $\vec{a} \cdot \vec{F}$:

$$\vec{a} \cdot \vec{F} = 0{,}02 \cdot 0{,}04 \cdot \cos 45^0 = 0{,}0008 \cdot \frac{\sqrt{2}}{2} = 0{,}0004 \cdot \sqrt{2},$$

thus, a force work is $A = 0{,}0004 \cdot \sqrt{2}$ (kg/m).

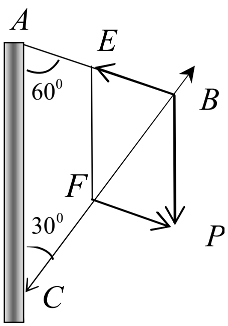

Figure 29

Example 23. Determine the force in the bars AB and BC if the load $P = 3H$ is suspended so on these bars as shown in the figure 29.

Solution. The load P create a force $\vec{F}$ which, as we know from mechanics, balances the reaction of the bars AB and BC, and besides, the reaction of the bar AB is directed along this bar from the point B to the point A, and the reaction of the bar BC is directed along this bar from the point C to the point B (Appendix B). Denote the reaction of the bars AB and BC, respectively, as $\vec{T_1}$ and $\vec{T_2}$.

Then from the task we get

$$\vec{F} = \vec{T_1} + \vec{T_2}.$$

Construct a parallelogram $EBPF$. Since $\vec{F} = \overrightarrow{EF}$, $\vec{T_1} = \overrightarrow{BE}$, $\vec{T_2} = \overrightarrow{FB}$. Then, considering equality $\vec{F} = \vec{T_1} + \vec{T_2}$, we get:

$$\overrightarrow{EF} = \overrightarrow{BE} + \overrightarrow{FB}.$$

From the task we know that $\left|\overrightarrow{EF}\right| = 3$, $\angle EFB = 30^0$, $\angle BEF = 60^0$, and consequently $\angle FBE = 90^0$. Thus, from a right triangle we have the followings:

$$\left|\overrightarrow{EB}\right| = \left|\overrightarrow{EF}\right| \cdot \sin 30^0 = 3 \cdot \frac{1}{2} = \frac{3}{2} = 1{,}5;$$

$$\left|\overrightarrow{FB}\right| = \left|\overrightarrow{EF}\right| \cdot \cos 30^0 = 3 \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} = 1{,}5 \cdot \sqrt{3};$$

from here we get that $\left|\overrightarrow{T_1}\right| = 1{,}5H$, $\left|\overrightarrow{T_2}\right| = 1{,}5\cdot\sqrt{3}H$. Thereby, the force in the bar AB equals $1{,}5H$, and the force in the bar BC equals $1{,}5\cdot\sqrt{3}H$.

Example 24. Force $\vec{F} = \dot{i} - 8\vec{j}$ is spread out in two directions, one of them is given $\overrightarrow{F_1} = 2\dot{i} + 2\vec{j}$. Find the second component of the force $\vec{F}$.

Solution. The unknown component of force $\vec{F}$ is denoted by $\overrightarrow{F_2} = (x, y)$. Since $\vec{F} = \overrightarrow{F_1} + \overrightarrow{F_2}$ then we have

$$\vec{F} = \overrightarrow{F_1} + \overrightarrow{F_2} = 2\dot{i} + 2\vec{j} + x\dot{i} + y\vec{j},\ \dot{i} - 8\vec{j} = 2\dot{i} + 2\vec{j} + x\dot{i} + y\vec{j}$$

or

$$\dot{i} - 8\vec{j} = \dot{i}(x+2) + \vec{j}(y+2),\ \dot{i}(x+2-1) + \vec{j}(y+2+8) = 0.$$

Since the vectors $\dot{i}$ and $\vec{j}$ are not equal to zero at the same time, equality could be equal zero only if the coefficients before $\dot{i}$ and $\vec{j}$ are simultaneous equal to zero, i.e.

$$x + 2 - 1 = 0 \text{ and } y + 2 + 8 = 0,$$

solving these equations together we get the answer: $x = -1$, $y = -10$. In this way, the second component of the force $\vec{F}$ is $\overrightarrow{F_2} = -\dot{i} - 10\vec{j}$.

Example 25. Find a vector $\vec{d}$ is collinear to vector $\vec{a} = (1{,}2{,}-3)$ and satisfies the condition $\vec{a}\cdot\vec{d} = 28$.

Solution. Let vector $\vec{d}$ has coordinates (x, y, z), so $\vec{d}$ is collinear to vector $\vec{a}$ and $\vec{d} = \lambda\vec{a}$ according to the definition of collinear vectors. Then $x = \lambda x_{\vec{a}}$, $y = \lambda y_{\vec{a}}$, $z = \lambda z_{\vec{a}}$, i.e. $x = \lambda$, $y = 2\lambda$, $z = -3\lambda$. Also we know that the scalar product of vectors $\vec{d}$ and $\vec{a}$ equals 28 which can be calculated by the following formula

$$\vec{a}\cdot\vec{d} = x_1 \cdot x_2 + y_1 \cdot y_2 + z_1 \cdot z_2 .$$

In our tasks it looks like

$$\vec{a}\cdot\vec{d} = 1\cdot\lambda + 2\cdot 2\lambda - 3\cdot(-3\lambda),\ \vec{a}\cdot\vec{d} = 28,\ \lambda + 4\lambda + 9\lambda = 28,$$

$$14\lambda = 28,\ \lambda = 28:14,\ \lambda = 2 .$$

So we found out the proportionality coefficient λ and now we can find the coordinates of the sought vector $\vec{d}$ directly, that is

$$x = 2,\ y = 4,\ z = -6;\ \vec{d} = (2,4,-6).$$

Example 26. Find a vector $\vec{p}$ knowing that it is perpendicular to the vectors $\vec{a} = (2,-3,1)$, $\vec{b} = (-1,-2,3)$ and satisfies the condition $\vec{p}\left(\vec{i} + 2\vec{j} - 7\vec{k}\right) = 16$.

Solution. Denote the coordinates of unknown vector $\vec{p}$ as (x, y, z). By the task, this vector $\vec{p}$ is perpendicular to the two vectors $\vec{a}$ and $\vec{b}$, and, therefore, the scalar products of these vectors are equal to zero, i.e. $\vec{a}\cdot\vec{p} = 0$, $\vec{b}\cdot\vec{p} = 0$, According to the formula for calculating the scalar product in the coordinate form, we get:

$$\vec{a}\cdot\vec{p} = 0,\ 2x - 3y + z = 0;$$

$$\vec{b}\cdot\vec{p} = 0,\ -x - 2y + 3z = 0 .$$

Also, we know about third scalar product, namely, it is $\vec{p}\left(\vec{i} + 2\vec{j} - 7\vec{k}\right) = 16$ or $x + 2y - 7z = 16$.

Taking into account all three conditions, we compose a system of equations and solve it using the Cramer method.

$$\begin{cases} 2x - 3y + z = 0, \\ -x - 2y + 3z = 0, \\ x + 2y - 7z = 16. \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & -3 & 1 \\ -1 & -2 & 3 \\ 1 & 2 & -7 \end{vmatrix} = 28 - 9 - 2 + 2 + 21 - 12 = 28,$$

$$\Delta_1 = \begin{vmatrix} 0 & -3 & 1 \\ 0 & -2 & 3 \\ 16 & 2 & -7 \end{vmatrix} = (-1)^{3+1} \cdot 16 \cdot (-9 + 2) = -112,$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-112}{28} = -4;$$

$$\Delta_2 = \begin{vmatrix} 2 & 0 & 1 \\ -1 & 0 & 3 \\ 1 & 16 & -7 \end{vmatrix} = (-1)^{3+2} \cdot 16 \cdot (6 + 1) = -112,$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-112}{28} = -4;$$

$$\Delta_3 = \begin{vmatrix} 2 & -3 & 0 \\ -1 & -2 & 0 \\ 1 & 2 & 16 \end{vmatrix} = (-1)^{3+3} \cdot 16 \cdot (-4 - 3) = -112,$$

$$z = \frac{\Delta_2}{\Delta} = \frac{-112}{28} = -4.$$

Check the obtained results, substituting the found values of variables $x = -4$, $y = -4$, $z = -4$ into the each equation of the system

$$\begin{cases} 2\cdot(-4)-3\cdot(-4)-4=0, \\ 4-2\cdot(-4)+3\cdot(-4)=0, \\ -4+2\cdot(-4)-7\cdot(-4)=16; \end{cases} \begin{cases} -8+12-4=0, \\ 4+8-12=0, \\ -4-8+28=16; \end{cases} \begin{cases} 0=0, \\ 0=0, \\ 16=16. \end{cases}$$

So, the vector $\vec{p}$ satisfying the given task conditions has the coordinates $x=-4$, $y=-4$, $z=-4$.

Example 27. Prove that a quadrilateral with vertices at points $A(-5,3,4)$, $B(-1,-7,5)$, $C(6,-5,-3)$, $D(2,5,-4)$ is a square.

Solution. Define the coordinates of the vectors:

$$\overrightarrow{AB}=(-1+5,-7-3,5-4)=(4,-10,1);$$
$$\overrightarrow{BC}=(6+1,-5+7,-3-5)=(7,2,-8);$$
$$\overrightarrow{DC}=(2-6,5+5,-4+3)=(4.-10,1);$$
$$\overrightarrow{AD}=(2+5,5-3,-4-4)=(7,2,-8).$$

Comparing the vectors coordinates, we get

$$\overrightarrow{AB}=\overrightarrow{DC},\ \overrightarrow{BC}=\overrightarrow{AD}.$$

Since

$$\left|\overrightarrow{AB}\right|=\sqrt{4^2+(-10)^2+1^2}=\sqrt{117},\ \left|\overrightarrow{BC}\right|=\sqrt{7^2+3^2+8^2}=\sqrt{117},$$

then

$$\left|\overrightarrow{AB}\right|=\left|\overrightarrow{AD}\right|=\left|\overrightarrow{BC}\right|=\left|\overrightarrow{DC}\right|.$$

Consider the scalar product

$$\overrightarrow{AB}\cdot\overrightarrow{BC}=4\cdot7+(-10)\cdot2+1\cdot(-8)=0\Rightarrow\overrightarrow{AB}\perp\overrightarrow{BC}.$$

Therefore a quadrangle $ABCD$ is a square.

VECTOR OR CROSS PRODUCT

Definition 17. The *vector product* (or *cross product*) of vectors $\vec{a}$ and the non-collinear to it vector $\vec{b}$ is the third vector $\vec{c}$ which is satisfied to the following conditions:

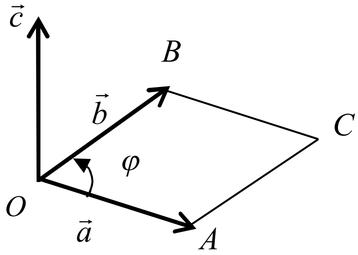

Figure 30

1) vector $\vec{c}$ (Figure 30) is perpendicular to two vectors $\vec{a}$ and $\vec{b}$;

2) the magnitude of the vector $\vec{c}$ is equal to the product of magnitudes of the vectors $\vec{a}$ and $\vec{b}$ and the cosine of the angel between them, i.e.

$$\left|\vec{c}\right| = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \sin\varphi ;$$

3) coordinates systems of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{i}$, $\vec{j}$, $\vec{k}$ have the same orientation (Appendix C) and turning from the first vector to the second one have to be a moving in the direction of counter clock-wise.

The vector product is denoted $\left\lfloor \vec{a}, \vec{b} \right\rfloor$ or $\vec{c} = \vec{a} \times \vec{b}$.

Note 19. The result of vector product is a vector.

Geometrical interpretation of a vector product

According to the vector product definition, the magnitude of the vector $\vec{c}$ ($\left|\vec{c}\right| = \left|\vec{a} \times \vec{b}\right|$) equals area S of the parallelogram $OACB$ (Figure 30) built on these vectors $\vec{a}$ and $\vec{b}$.

Example 28. Calculate the area of the parallelogram built on the given vectors $\vec{a}$ and $\vec{b}$, if their magnitudes are $0{,}8$ and $0{,}5$ respectively, and the angel between them is 30^0.

Solution. From geometric interpretation of vector product we know that the parallelogram area equals the magnitude of the vector product, $S = \left|\vec{a} \times \vec{b}\right|$ or $S = \left|\vec{c}\right|$, also $\left|\vec{c}\right| = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \sin\varphi$, so we can find the area

$$S = \left|\vec{c}\right| = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \sin\varphi = 0{,}8 \cdot 0{,}5 \cdot \sin 30^0 = 0{,}2 \text{ (units of area).}$$

Note 20. Geometrically, the vector product is useful as a method for constructing a vector perpendicular to a plane if you have two vectors in this plane.

Note 21. If vectors $\vec{a}$ and $\vec{b}$ are collinear then the parallelogram $OACB$ has a zero area. In accordance with it, the vector product of two collinear vectors is zero vector, $\vec{0}$. And we have a *collinearity condition* as

$$\vec{a} // \vec{b} \Leftrightarrow \vec{a} \times \vec{b} = \vec{0}.$$

Physical interpretation of vector product

Let we have a system of material points *A*, *C*, *B* (Figure 31) which are bonded to each other and the point O. Let a force $\vec{F_1}$ be applied at the point A, a force $\vec{F_2}$ be applied at the point B, a force $\vec{F_3}$ be applied at the point C. The physical value a *moment* $\vec{M_1}$ *of the force* $\vec{F_1}$ (Appendix D) applied at the point A relatively to the point O is drawing by the vector product $\vec{OA} \times \vec{F_1}$. Similarly, *moments* $\vec{M_2}$, $\vec{M_3}$ *of the forces* $\vec{F_2}$ and $\vec{F_3}$ can be written as $\vec{OB} \times \vec{F_2}$, $\vec{OC} \times \vec{F_3}$.

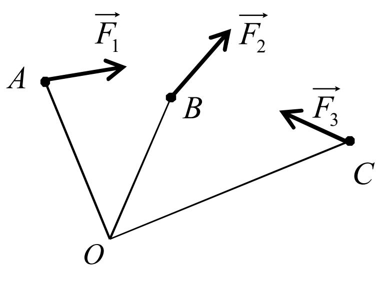

Figure 31

If a sum of these vectors

$$\vec{S}=\overrightarrow{OA}\times\vec{F}_1+\overrightarrow{OB}\times\vec{F}_2+\overrightarrow{OC}\times\vec{F}_3$$

is a zero vector, then a system is static equilibrium; if the vector $\vec{S}$ is not zero vector, the system is in the rotational motion.

Properties of the vector product

1. The vector product is not commutative, it means the order in which we do the calculation does matter and changing items order in the vector product leads to the changing of its sign

$$\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}.$$

2. The vector product is distributive over addition:

$$\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c} \text{ or } \left(\vec{a}+\vec{b}\right)\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}.$$

3. Associative property with respect to a scalar factor:

$$\left(\lambda\vec{a}\right)\times\vec{b}=\lambda\left(\vec{a}\times\vec{b}\right)$$

4. The vector product of the vector on itself (or two equal vectors) is zero vector:

$$\vec{a}\times\vec{a}=\vec{0}.$$

Example 29. Simplify the expression $\left(2\vec{a}-3\vec{b}\right)\times\left(\vec{a}+5\vec{b}\right)$.

Solution. Open the parentheses, considering the given above properties of the vector product, and get

$$\left(2\vec{a}-3\vec{b}\right)\times\left(\vec{a}+5\vec{b}\right)=2\vec{a}\times\vec{a}-3\vec{b}\times\vec{a}+2\vec{a}\times5\vec{b}-3\vec{b}\times5\vec{b}=$$

$$=\vec{0}-3\vec{b}\times\vec{a}+2\vec{a}\times5\vec{b}-\vec{0}=3\vec{a}\times\vec{b}+10\vec{a}\times\vec{b}=13\vec{a}\times\vec{b}.$$

5. The vector product of the unit vectors (Appendix A):

$$\vec{i}\times\vec{i}=\vec{0},\ \vec{j}\times\vec{j}=\vec{0},\ \vec{k}\times\vec{k}=\vec{0};$$

$$\vec{i}\times\vec{j}=\vec{k}\ (\vec{k}\perp\vec{i},\vec{k}\perp\vec{j}),\ \ \vec{j}\times\vec{k}=\vec{i}\ (\vec{j}\perp\vec{i},\vec{k}\perp\vec{i}),$$

$$\vec{i}\times\vec{k}=-\vec{j}\ (\vec{j}\perp\vec{i},\vec{k}\perp\vec{j}).$$

Example 30. Calculate the triangular area on the vectors $\vec{p} = \vec{a} - 2\vec{b}$, $\vec{q} = 2\vec{a} + 2\vec{b}$, if $|\vec{a}| = |\vec{b}| = 3$, and angel $\angle(\vec{a}, \vec{b}) = 60^0$.

Solution. In accordance with geometrical interpretation of the vector product, the parallelogram area can be found as a magnitude of the vector product, thus triangular area can be computed as half parallelogram area, namely,

$$S_\Delta = \frac{1}{2}\left|\vec{p} \times \vec{q}\right|.$$

Find the vector product magnitude of the vectors $\vec{p}$ and $\vec{q}$:

$$\vec{p} \times \vec{q} = \left(\vec{a} - 2\vec{b}\right) \times \left(2\vec{a} + 2\vec{b}\right) = 2\vec{a} \times \vec{a} - 4\vec{b} \times \vec{a} + \vec{a} \times 2\vec{b} - 4\vec{b} \times \vec{b} =$$

$$= -4\vec{b} \times \vec{a} + \vec{a} \times 2\vec{b} = 4\vec{a} \times \vec{b} + 2\vec{a} \times \vec{b} = 6\vec{a} \times \vec{b}; \; \left|\vec{p} \times \vec{q}\right| = \left|6\vec{a} \times \vec{b}\right|,$$

$$6\left|\vec{a} \times \vec{b}\right| = 6|\vec{a}||\vec{b}| \sin \angle\left(\vec{a}, \vec{b}\right) = 6 \cdot 3 \cdot 3 \cdot \sin 60^0 = 54 \cdot \frac{\sqrt{3}}{2} = 27\sqrt{3}.$$

So, the desired triangular area is $27\sqrt{3}$ (units of area).

Vector product in Cartesian form

If we need to find the vector product of the vectors given in a coordinate form, like this $\vec{a} = (x_1, y_1, z_1)$ and $\vec{b} = (x_2, y_2, z_2)$, we can use the determinant with these vectors coordinates and the first row is the standard basis vectors (unit vectors) and must appear in the order given here. It looks like this

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix},$$

or, coordinates of vector $\vec{c}$, which is the vector product,

$$\vec{a}\times\vec{b}=\vec{c}=\left\{\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right\}.$$

Also, the determinant can be calculated using the method of cofactors, as the formula is below:

$$\vec{a}\times\vec{b}=\vec{i}\cdot\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}-\vec{j}\cdot\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}+\vec{k}\cdot\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

Example 31. Three forces $\overrightarrow{F_1}=(2,4,6)$, $\overrightarrow{F_2}=(1,-2,3)$, $\overrightarrow{F_3}=(1,1,-7)$ applied to the point $A(3,-4,8)$. Determine the magnitude and direction cosines of the resultant force moment of these forces relative to the point $B(4,-2,6)$.

Solution. If $\vec{F}$ is a force applied at the point A and vector $\vec{a}=\overrightarrow{BA}$, then a vector $\vec{a}\times\vec{F}$ is a moment of the force $\vec{F}$ relative to the point B.

Find the resultant force of three given forces:

$$\vec{F}=\overrightarrow{F_1}+\overrightarrow{F_2}+\overrightarrow{F_3},$$

i.e. it is a sum of three vectors:

$$\vec{F}=(2+1+1,4-2+1,6+3-7)=(4,3,2).$$

Having the coordinates of start point and end point, find the coordinates of vector $\vec{a}=\overrightarrow{BA}$:

$$\vec{a}=\overrightarrow{BA}=(3-4,-2-4,8-6)=(-1,-2,2).$$

Use the vector product formula in Cartesian form to find the vector product $\vec{a}\times\vec{F}$:

$$\vec{a}\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & -2 & 2\\ 4 & 3 & 2\end{vmatrix}=-4\vec{i}+8\vec{j}-3\vec{k}+8\vec{k}+2\vec{j}-6\vec{i}=-10\vec{i}+10\vec{j}+5\vec{k},$$

$$\vec{a}\times\vec{F}=(-10,10,5).$$

And a value of the resultant force moment is

$$\left|\vec{a}\times\vec{F}\right|=\sqrt{10^2+10^2+5^2}=\sqrt{225}=15.$$

Its direction cosines are

$$\cos\alpha=\frac{-10}{15}=\frac{-2}{3},\ \cos\beta=\frac{10}{15}=\frac{2}{3},\ \cos\gamma=\frac{5}{15}=\frac{1}{3}.$$

Example 32. Prove, that $\left(\vec{a}\times\vec{b}\right)^2+\left(\vec{a}\cdot\vec{b}\right)^2=\vec{a}^{\,2}\cdot\vec{b}^{\,2}$.

Solution. Use definitions of a scalar product and vector product of vectors

$$\vec{a}\times\vec{b}=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\sin\left(\widehat{\vec{a},\vec{b}}\right);\ \vec{a}\cdot\vec{b}=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\cos\left(\widehat{\vec{a},\vec{b}}\right),$$

square both sides of each of the written equalities:

$$\left(\vec{a}\times\vec{b}\right)^2=\left|\vec{a}\right|^2\cdot\left|\vec{b}\right|^2\cdot\sin^2\left(\widehat{\vec{a},\vec{b}}\right);\ \left(\vec{a}\cdot\vec{b}\right)^2=\left|\vec{a}\right|^2\cdot\left|\vec{b}\right|^2\cdot\cos^2\left(\widehat{\vec{a},\vec{b}}\right),$$

and add them

$$\left(\vec{a}\times\vec{b}\right)^2+\left(\vec{a}\cdot\vec{b}\right)^2=\left|\vec{a}\right|^2\cdot\left|\vec{b}\right|^2\cdot\sin^2\left(\widehat{\vec{a},\vec{b}}\right)+\left|\vec{a}\right|^2\cdot\left|\vec{b}\right|^2\cdot\cos^2\left(\widehat{\vec{a},\vec{b}}\right)=$$

$$=\left|\vec{a}\right|^2\cdot\left|\vec{b}\right|^2\cdot\left[\sin^2\left(\widehat{\vec{a},\vec{b}}\right)+\cos^2\left(\widehat{\vec{a},\vec{b}}\right)\right]=\left|\vec{a}\right|^2\cdot\left|\vec{b}\right|^2.$$

Since, the magnitude square of the vector is equal to its scalar square, i.e. $|\vec{a}|^2 = \vec{a}^2$ and $|\vec{b}|^2 = \vec{b}^2$, the resulting equality can be represented

$$\left(\vec{a}\times\vec{b}\right)^2 + \left(\vec{a}\cdot\vec{b}\right)^2 = \vec{a}^2\cdot\vec{b}^2 .$$

Example 33. Find the angle between the diagonals of a parallelogram built on vectors $\vec{a} = \vec{p} - 3\vec{q}$, $\vec{b} = 2\vec{p} - 4\vec{q}$ if $|\vec{p}| = |\vec{q}| = 1$, $\left(\vec{p}\,\hat{,}\,\vec{q}\right) = 60^0$.

Solution. As you know, to calculate a parallelogram area, we can use these formulas

$$S = \left|\vec{a}\times\vec{b}\right| \text{ or } S = \left|\vec{d_1}\right|\cdot\left|\vec{d_1}\right|\cdot\sin\left(\vec{d_1}\,\hat{,}\,\vec{d_2}\right).$$

From the last formula we can find the sine of the angle between diagonals,

$$\sin\left(\vec{d_1}\,\hat{,}\,\vec{d_2}\right) = \frac{S}{\left|\vec{d_1}\right|\cdot\left|\vec{d_1}\right|},$$

but, for this, we need to know the area of the parallelogram and the lengths of its diagonals. Applying the parallelogram rule for adding and subtracting vectors, we can find the lengths of the diagonals. Since we know that one of its diagonals is the sum of the vectors on which it is built, and the other is the subtraction.

$$\vec{d_1} = \vec{a} + \vec{b} = \vec{p} - 3\vec{q} + 2\vec{p} - 4\vec{q} = 3\vec{p} - 7\vec{q},$$

$$\vec{d_2} = \vec{a} - \vec{b} = \vec{p} - 3\vec{q} - 2\vec{p} + 4\vec{q} = -\vec{p} + \vec{q}.$$

Calculate the magnitudes of these vectors

$$\left|\vec{d_1}\right| = \sqrt{\left|\vec{d_1}\right|^2} = \sqrt{\left(3\vec{p} - 7\vec{q}\right)^2} = \sqrt{9\vec{p}^2 - 42\vec{p}\cdot\vec{q} + 49\vec{q}^2} =$$

$$= \sqrt{9\left|\vec{p}\right|^2 - 42\left|\vec{p}\right|\cdot\left|\vec{q}\right|\cdot\cos\left(\widehat{\vec{p},\vec{q}}\right) + 49\left|\vec{q}\right|^2} = \sqrt{9 - 42\cdot\frac{1}{2} + 49} = \sqrt{37}\ ;$$

$$\left|\vec{d_2}\right| = \sqrt{\left|\vec{d_2}\right|^2} = \sqrt{\left(-\vec{p} + \vec{q}\right)^2} = \sqrt{\vec{p}^2 - 2\vec{p}\cdot\vec{q} + \vec{q}^2} =$$

$$= \sqrt{\left|\vec{p}\right|^2 - 2\left|\vec{p}\right|\cdot\left|\vec{q}\right|\cdot\cos\left(\widehat{\vec{p},\vec{q}}\right) + \left|\vec{q}\right|^2} = \sqrt{1 - 2\cdot\frac{1}{2} + 1} = 1\,.$$

Calculate the parallelogram area

$$S = \left|\vec{a}\times\vec{b}\right| = \left|\left(\vec{p} - 3\vec{q}\right)\times\left(\vec{q} - \vec{p}\right)\right| = \left|\vec{p}\times\vec{q} - \vec{p}\times\vec{p} - 3\vec{q}\times\vec{q} + 3\vec{q}\times\vec{p}\right|,$$

considering these properties of a cross product of the vectors $\vec{p}$ and $\vec{q}$: $\vec{p}\times\vec{p} = 0$, $3\vec{q}\times\vec{q} = 0$ and $\vec{p}\times\vec{q} = -\vec{p}\times\vec{q}$, we get

$$S = \left|-2\vec{p}\times\vec{q}\right| = 2\left|\vec{p}\times\vec{q}\right| = 2\left|\vec{p}\right|\cdot\left|\vec{q}\right|\sin\left(\widehat{\vec{p},\vec{q}}\right) = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}\,.$$

Now we can calculate the angle between the diagonals of a parallelogram

$$\sin\left(\widehat{\vec{d_1},\vec{d_2}}\right) = \frac{\sqrt{3}}{\sqrt{31}} = \sqrt{\frac{3}{31}} \approx 0{,}0810\,,$$

$$\left(\widehat{\vec{d_1},\vec{d_2}}\right) = \arcsin 0{,}0810\,.$$

Use the special tables or engineering calculator to find the angle.

Example 34. Find a vector that is orthogonal to the plane containing the points $A(3,-1,8)$, $B(4,-2,1)$, $C(5,3,-1)$

Solution. First of all we need two vectors that are both parallel to the plane. Using the points that we have (all in the plane) we can quickly get quite a few vectors that are parallel to the plane. Use the following two vectors.

$$\overrightarrow{AB}=(4-3,-2+1,1-8)=(1,-1,-7);$$

$$\overrightarrow{AC}=(5-3,3+1,-1-8)=(2,4,-7).$$

Now we know that the cross product of any two vectors will be orthogonal to the two original vectors. Since the two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel to the plane (actually, they are lied in the plane in this case!), also, as we know that the cross product must then be orthogonal, or normal, to the plane. So, using the "trick" we used in the notes the cross product is,

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & -7\\ 2 & 4 & -7\end{vmatrix}=7\vec{i}-14\vec{j}+4\vec{k}+2\vec{k}+7\vec{j}+28\vec{i}=35\vec{i}-7\vec{j}+6\vec{k}.$$

So, the desired orthogonal plane vector is $\vec{a}=(35,-7,6)$.

Example 35. Are vectors $\vec{a}=(2,-1,-3)$ and $\vec{b}=(-6,3,9)$ parallel?

Solution. Use the cross product

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -1 & -3\\ -6 & 3 & 9\end{vmatrix}=-9\vec{i}+18\vec{j}+6\vec{k}-6\vec{k}-18\vec{j}+9\vec{i}=\vec{0}.$$

So, the given vectors are parallel.

SCALAR TRIPLE PRODUCT

Definition 15. The *scalar triple product* **(**also called the *mixed product*, *box product*, or *triple scalar product*) of three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is a number equals the scalar product of the vector $\vec{c}$ with the vector product of two vectors, $\vec{a}\times\vec{b}$.

It can be written $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$ or $\vec{a}\vec{b}\vec{c}$. Here, the parentheses may be omitted without causing ambiguity, since the scalar product cannot be evaluated first. If it were, it would leave the vector product of a scalar and a vector, which is not defined.

Note that these three vectors are not coplanar (look at definition 9). Case, when these vectors are coplanar, will be considered further.

The result of the scalar triple product is a number that can be as positive as negative, it depends on an orientation of a vectors system $\vec{a}$, $\vec{b}$, $\vec{c}$. If this system has a right-hand orientation then the sign of the scalar triple product is plus, otherwise, it is minus sign.

Geometrical interpretation of scalar triple product

The geometrical interpretation of the scalar triple product is the (signed) volume of the parallelepiped (Figure 32) defined by the three given vectors $\vec{a}$, $\vec{b}$, $\vec{c}$:

$$V=\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|=\left|\vec{a}\vec{b}\vec{c}\right|.$$

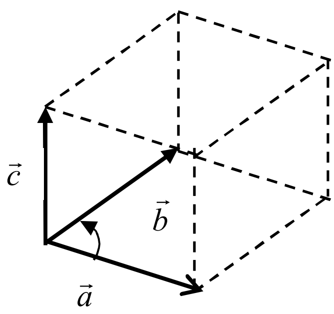

Figure 32

In case of three coplanar vectors, the building of a parallelepiped on these vectors becomes impossible thus its volume equals zero, and the *coplanar condition* can be formulated as: if three vectors are coplanar then their scalar triple product is zero,

$$\vec{a}\vec{b}\vec{c} = 0 .$$

Properties of the scalar triple product

1. The scalar triple product is unchanged under a circular shift of its three items

$$\vec{a}\vec{b}\vec{c} = \vec{c}\vec{a}\vec{b} = \vec{b}\vec{c}\vec{a} .$$

2. Swapping the positions of the operators without re-ordering the operands leaves the triple product unchanged. This follows from the preceding property and the commutative property of the scalar product

$$\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} = \vec{a} \cdot \left(\vec{b} \times \vec{c}\right).$$

3. Swapping any two of the three items negates the triple product. This follows from the circular-shift property and the non-commutative of the vector product

$$\vec{a}\vec{b}\vec{c} = -\vec{b}\vec{a}\vec{c} = -\vec{a}\vec{c}\vec{b} = -\vec{c}\vec{b}\vec{a} .$$

4. Distribution property

$$\vec{a} \cdot \left(\vec{b} + \vec{d}\right) \cdot \vec{c} = \vec{a}\vec{b}\vec{c} + \vec{a}\vec{d}\vec{c} .$$

5. Associative property relative to a numerical factor

$$\lambda\left(\vec{a}\vec{b}\vec{c}\right) = \vec{a}\left(\lambda\vec{b}\right)\vec{c} = \vec{a}\vec{b}\left(\lambda\vec{c}\right).$$

If vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are given in Cartesian form, $\vec{a} = (x_1, y_1, z_1)$, $\vec{b} = (x_2, y_2, z_2)$ and $\vec{c} = (x_3, y_3, z_3)$ then their scalar triple product is calculated by formula

$$\vec{a}\vec{b}\vec{c} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix},$$

and a volume of the parallelepiped defined by these three vectors is

$$V = \left|\vec{a}\vec{b}\vec{c}\right| = \pm\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix},$$

in which the sign is taken identical with the sign of the determinant.

Example 36. Vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are satisfied to the state $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0$. Prove that these vectors are coplanar.

Solution. Do the scalar product with vectors $\vec{a}$ and the given state and get

$$\vec{a}\left(\vec{a}\times\vec{b}\right)+\vec{a}\left(\vec{b}\times\vec{c}\right)+\vec{a}\left(\vec{c}\times\vec{a}\right)=\vec{a}\cdot 0 \text{ or } \vec{a}\vec{a}\vec{b}+\vec{a}\vec{b}\vec{c}+\vec{a}\vec{c}\vec{a}=0.$$

However, vectors $\vec{a}$, $\vec{a}$, $\vec{b}$ and $\vec{a}$, $\vec{c}$, $\vec{a}$ are coplanar (according to the definition), come to the conclusion

$$\vec{a}\vec{a}\vec{b}=0,\ \vec{a}\vec{c}\vec{a}=0,$$

and the expression $\vec{a}\vec{a}\vec{b}+\vec{a}\vec{b}\vec{c}+\vec{a}\vec{c}\vec{a}=0$ can be changed by the new $\vec{a}\vec{b}\vec{c}=0$, based on which we can say that the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar.

Note 21. The scalar triple product, where two vectors are equal, is zero.

Example 37. Determine the parameter value λ when vectors $\vec{a}=(1,2,-2)$, $\vec{b}=(1,-2,1)$, $\vec{c}=(\lambda,-2,-1)$ are coplanar.

Solution. Calculate the scalar triple product

$$\vec{a}\vec{b}\vec{c} = \begin{vmatrix} 1 & 2 & -2 \\ 1 & -2 & 1 \\ \lambda & -2 & -1 \end{vmatrix} = 2 + 2\lambda + 4 - 4\lambda + 2 + 2 = 10 - 2\lambda .$$

If the given vectors are coplanar then $\vec{a}\vec{b}\vec{c} = 0$, so

$$10 - 2\lambda = 0 \Rightarrow \lambda = 5 .$$

Therefore, if $\lambda = 5$ the given vectors are coplanar.

Example 38. Calculate the volume of pyramid (tetrahedron) $ABCD$ (Figure 33) if points coordinates $A(2,2,1)$, $B(3,0,3)$, $C(13,4,11)$, $D(0,2,5)$ are given.

Solution. Calculate the volume of pyramid $ABCD$ using a formula:

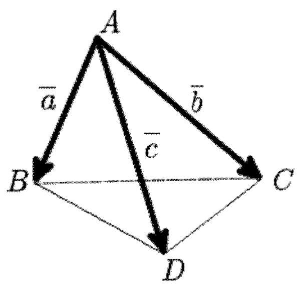

Figure 33

$$V = \frac{1}{6}\left|(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD})\right| .$$

First of all, find the coordinates of vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ starting from a common point A:

$$\overrightarrow{AB} = (3-2, 0-2, 3-1) = (1, -2, 2),$$

$$\overrightarrow{AC} = (13-2, 4-2, 11-1) = (11, 2, 10),$$

$$\overrightarrow{AD} = (0-2, 2-2, 5-1) = (-2, 0, 4);$$

$$(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}) = \begin{vmatrix} 1 & -2 & 2 \\ 11 & 2 & 10 \\ -2 & 0 & 4 \end{vmatrix} = 8 + 0 + 40 - (-8) -$$

$$-(-88) - 0 = 144, \quad V = \frac{1}{6}|144| = 24 \text{ (units of volume).}$$

Example 39. Pyramid vertices are $A_1(2,-3,5)$, $A_2(0,2,1)$, $A_3(-2,-2,3)$, $A_4(3,2,4)$. Calculate the length of the height drawn from the vertex A_4.

Solution. The volume of pyramid can be found by the formula, $V=\frac{S_{base}\cdot H}{3}$, whence $H=\frac{3V}{S_{base}}$.

Base $A_1A_2A_3$ is a triangle; find his area using the vector product:

$$S_{\Delta A_1A_2A_3}=\frac{1}{2}\left|\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}\right|,\ \overrightarrow{A_1A_2}=(0-2,2+3,1-5)=(-2,5,-4),$$

$$\overrightarrow{A_1A_3}=(-2-2,-2+3,3-5)=(-4,1,-2),$$

$$\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -2 & 5 & -4\\ -4 & 1 & -2\end{vmatrix}=\vec{i}\begin{vmatrix}5 & -4\\ 1 & -2\end{vmatrix}-\vec{j}\begin{vmatrix}-2 & -4\\ -4 & -2\end{vmatrix}+\vec{k}\begin{vmatrix}-2 & 5\\ -4 & 1\end{vmatrix}=$$

$$=(-10+4)\vec{i}-(4-16)\vec{j}+(-2+20)\vec{k}=-6\vec{i}+12\vec{j}+18\vec{k},$$

$$\left|\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}\right|=\sqrt{(-6)^2+12^2+18^2}=\sqrt{504}=6\sqrt{14},$$

$$S_{\Delta A_1A_2A_3}=\frac{1}{2}\cdot 6\sqrt{14}=3\sqrt{14}\ \text{(unit of area)}.$$

Calculate the volume of pyramid $A_1A_2A_3A_4$ using the scalar triple product of three vectors defined this pyramid. It is vectors $\overrightarrow{A_1A_2}$, $\overrightarrow{A_1A_3}$, $\overrightarrow{A_1A_4}$:

$$V=\frac{1}{6}\left|(\overrightarrow{A_1A_2},\overrightarrow{A_1A_3},\overrightarrow{A_1A_4})\right|,\ \overrightarrow{A_1A_4}=(3-2,2+3,4-5)=(1,5,-1),$$

$$(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}, \overrightarrow{A_1A_4}) = \begin{vmatrix} -2 & 5 & -4 \\ -4 & 1 & -2 \\ 1 & 5 & -1 \end{vmatrix} = 2 + 80 - 10 + 4 - 20 - 20 = 36,$$

$$V = \frac{1}{6}|36| = 6 \text{ (units of volume).}$$

Then $H = \dfrac{3 \cdot 6}{3\sqrt{14}} = \dfrac{6}{\sqrt{14}} = \dfrac{6 \cdot \sqrt{14}}{\sqrt{14} \cdot \sqrt{14}} = \dfrac{6 \cdot \sqrt{14}}{14} = \dfrac{3\sqrt{14}}{7}$.

Example 40. Prove that four points $A(1,2,-1)$, $B(0,1,5)$, $C(-1,2,1)$, $D(2,1,3)$ lie on the same plane.

Solution. As we know that four points lie on the same plane if three vectors formed by these points are coplanar, so their scalar triple product is zero.

Find the coordinates of these vectors, $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$:

$$\overrightarrow{AB} = (0-1, 1-2, 5+1) = (-1,-1,6),$$

$$\overrightarrow{AC} = (-1-1, 2-2, 1+1) = (-2,0,2),$$

$$\overrightarrow{AD} = (2-1, 1-2, 3+1) = (1,-1,4).$$

Calculate the scalar triple product:

$$\left(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right) = \begin{vmatrix} -1 & -1 & 6 \\ -2 & 0 & 2 \\ 1 & -1 & 4 \end{vmatrix} = 0 + 12 - 2 - 0 - 8 - 2 = 0.$$

Thereby, points $A(1,2,-1)$, $B(0,1,5)$, $C(-1,2,1)$, $D(2,1,3)$ lie on the same plane.

LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

Definition 16. Expression in the following form $\lambda_1\vec{a}+\lambda_2\vec{b}+\lambda_3\vec{c}+...+\lambda_n\vec{r}$ is called a *linear combination* of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$,..., $\vec{r}$ with coefficients λ_1, λ_2, …, λ_n which are arbitrary numbers.

Definition 17. The system of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$,..., $\vec{r}$ is called a *linearly dependent* if there is a linear combination equals zero vector:

$$\lambda_1\vec{a}+\lambda_2\vec{b}+\lambda_3\vec{c}+...+\lambda_n\vec{r}=0 \tag{1}$$

at least one of the coefficients λ_1, λ_2, …, λ_n in this linear combination is different from zero.

If vectors $\vec{a}$, $\vec{b}$, $\vec{c}$,..., $\vec{r}$ are linearly dependent, then at least one of them can be represented as a linear combination of the rest ones, for example: if $\lambda_1 \neq 0$ it follows from equality (1) that $\vec{a}=\mu_2\vec{b}+\mu_3\vec{c}+...+\mu_n\vec{r}$, where $\mu_i=-\lambda_i/\lambda_1$, $i=2,3,...,n$.

Definition 18. The system of vectors $\vec{a},\vec{b},\vec{c},...,\vec{r}$ is named *linearly independent* if equality (1) holds only for $\lambda_1=0$, $\lambda_2=0$, …, $\lambda_n=0$

Theorem (*the linear dependence of vector systems)*:

1) any four vectors in space are linearly dependent;

2) a system of three vectors is linearly dependent if and only if these vectors are coplanar;

3) a system of two vectors is linearly dependent if and only if these vectors are collinear;

4) a system of one vector is linearly dependent if and only if this vector is zero.

Definition 19. An ordered system of linearly independent vectors such that any vector can be represented as a linear combination of these vectors is named a *basis* in space.

A consequence of the theorem about the linear dependence of vectors systems:

1) in three-dimensional space a basis is an ordered triple of any non-coplanar vectors;

2) on the plane, the basis is any ordered pair of non-collinear vectors;

3) on a straight line, the basis is any non-zero vector;

4) in the space consisting of only the zero vector, the basis is not exists.

So, any vector can be represented as a linear combination of basis vectors. This representation (2) of the vector in the basis is called the *decomposition* and is carried out uniquely. The expansion coefficients are called *coordinates or components of a vector in the basis*. It can be written

$$\vec{a} = a_1\vec{e_1} + a_2\vec{e_2} + ... + a_n\vec{e_n}, \tag{2}$$

where $a_1, a_2, ..., a_n$ are coordinates of the vector $\vec{a}$ in the new basic.

Example 41. Prove that vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ form a basis. Find the coordinates of the vector $\vec{d}$ in this basis, if $\vec{a} = (1,3,6)$, $\vec{b} = (-3,4,-5)$, $\vec{c} = (1,-7,2)$, $\vec{d} = (-2,17,5)$.

Solution. As we know the three vectors form a basis if they are not coplanar. Let's make a determinant of the coordinates of these vectors and calculate it, If this determinant is not zero, then these vectors form a basis:

$$\begin{vmatrix} 1 & 3 & 6 \\ -3 & 4 & -5 \\ 1 & -7 & 2 \end{vmatrix} = 8 - 15 + 126 - 24 + 18 - 35 = 78 \neq 0,$$

that is, the given vectors form a basis.

Decompose a vector $\vec{d}$ in a new basis $\vec{a}, \vec{b}, \vec{c}$, write down the vector equation: $x\vec{a} + y\vec{b} + z\vec{c} = \vec{d}$.

Make a system of equations and solve it by Cramer's method:

$$\begin{cases} x - 3y + z = -2; \\ 3x + 4y - 7z = 17; \\ 6x - 5y + 2z = 5, \end{cases}$$

got the result that $\Delta = 78$,

$$\Delta_1 = \begin{vmatrix} -2 & -3 & 1 \\ 17 & 4 & -7 \\ 5 & -5 & 2 \end{vmatrix} = -16 - 85 + 105 - 20 + 102 + 70 = 156,$$

$$\Delta_2 = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 17 & -7 \\ 6 & 5 & 2 \end{vmatrix} = 34 + 15 + 84 - 102 + 12 + 35 = 78,$$

$$\Delta_3 = \begin{vmatrix} 1 & -3 & -2 \\ 3 & 4 & 17 \\ 6 & -5 & 5 \end{vmatrix} = 20 + 30 - 306 + 48 + 45 + 85 = -78,$$

$$x = \frac{156}{78} = 2, \quad y = \frac{78}{78} = 1, \quad z = \frac{-78}{78} = -1.$$

Check out: $$\begin{cases} 2 - 3 \cdot 1 - 1 = 2 - 3 - 1 = -2; \\ 3 \cdot 2 + 4 \cdot 1 - 7 \cdot (-1) = 6 + 4 + 7 = 17; \\ 6 \cdot 2 - 5 \cdot 1 + 2 \cdot (-1) = 12 - 5 - 2 = 5. \end{cases}$$

Thereby, $\vec{d} = 2\vec{a} + \vec{b} - \vec{c}$.

QUESTIONS FOR SELF-TESTS

1. What is a vector?
2. What is an opposite vector?
3. What is the vector magnitude? How do calculate it?
4. How to write the coordinates of a vector using unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$?
5. What are the direction cosines?
6. What conditions do you know about a point on the line?
7. How to find the coordinates of the point which is the middle of a given vector?
8. Is it possible to determine the coordinates of a point belonging to a straight line if we know the coordinates of two other points? Write down the formulas that can be used for this.
9. Formulate properties of a scalar product.
10. What do determine the sign of a scalar product?
11. In what interval is the angle between the vectors if their scalar product is positive?
12. How to use the concept of scalar product in mechanics?
13. What is a scalar square?
14. Does a scalar cube exist? Why?
15. What can you tell about the angle between the vectors if the projection of one vector onto another is negative?
16. Can we use transformations, which were applied in algebra, to ordinary pair products of factors of the first degree?
17. Can we use such transformations to the vector product?
18. What is a right-hand system?
19. Do you need to consider the orientation of a system?
20. In which case we have to consider the orientation of a system?
21. What is a geometric interpretation of the vector product?

22. Do the vector product commutative?

23. Can we use the vector product to find the vector perpendicular to the two vectors at the same time? Why?

24. In which way can we use the concept of a vector product to define a perpendicular vector?

25. How to calculate the triangular area?

26. Could items order be changed in the vector product?

27. What can you tell about coordinates of collinear vectors?

28. What is a moment of the force relative to the point?

29. What vectors do we call coplanar?

30. Which of the vectors products can be used to find the volume of the figure? How do compute it?

31. What is a coplanar condition?

32. What are the differences between the scalar product and scalar triple product? Do they have similarities?

33. What is a basis?

34. Which of vectors are called linear dependent?

35. Could any vector be represented as a linear combination of basis vectors? Why? In which form?

36. How to find the coordinates of the vector in a new basis?

TEST TO CONSOLIDATE THE STUDIED

Choose the correct answer.

1. Which of the following statements is the definition of a vector:
   a) quantity that has both magnitude and direction;
   b) quantity that has only magnitude;
   c) quantity that has only a direction;
   d) quantity that has a magnitude, but may or may not have a direction.
2. At the figure below find equal and opposite vectors

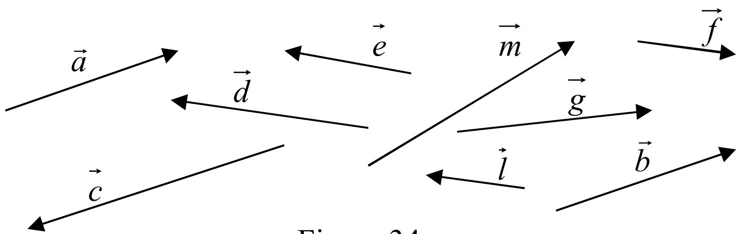

Figure 34

a) $\vec{a}=\vec{b}$, $\vec{f}=-\vec{l}$; b) $\vec{d}=\vec{e}$, $\vec{c}=-\vec{m}$;

c) $\vec{a}=\vec{g}$, $\vec{f}=-\vec{e}$; d) $\vec{e}=\vec{l}$, $\vec{f}=-\vec{d}$.

3. Which of the following statements is true:
   a) any two collinear vectors are linear dependent;
   b) three coplanar vectors are linear dependent;
   c) any two collinear vectors are linear independent;
   d) three coplanar vectors are linear dependent.
4. For what values of k is true the equality $proj_l k\vec{a} = k \cdot proj_l \vec{a}$:

   a) for any k; b) not for one; c) if $k>0$; d) if $k<0$.
5. Which of expressions determines the projection of the vector $\vec{a}$ onto the vector $\vec{b}$

a) $proj_{\vec{b}}\vec{a}=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$; b) $proj_{\vec{b}}\vec{a}=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$;

c) $proj_{\vec{b}}\vec{a}=\dfrac{|\vec{a}|}{\vec{a}\cdot\vec{b}}$; d) $proj_{\vec{b}}\vec{a}=\dfrac{|\vec{b}|}{\vec{a}\cdot\vec{b}}$

6. Define the collinear vectors: $\vec{a}=(0,3,5)$, $\vec{b}=(1,3,-2)$, $\vec{c}=(0,-3,-5)$, $\vec{d}=(-3,-9,6)$, $\vec{f}=(2,6,4)$, $\vec{e}=(3,9,6)$

a) $\vec{b}$ and $\vec{d}$; b) $\vec{a}$ and $\vec{c}$; c) $\vec{b}$ and $\vec{f}$; d) $\vec{b}$ and $\vec{e}$.

7. Which of these vectors: $\vec{a}=(0,3,5)$, $\vec{b}=(-1,0,5)$, $\vec{c}=(0,-2,0)$, $\vec{d}=(0,0,5)$, is parallel to the coordinate axis $0y$

a) $\vec{d}$; b) $\vec{c}$; c) $\vec{a}$; d) $\vec{b}$.

8. Express a vector $\overrightarrow{BE}$, which is a median of the triangle ABC, using vectors $\vec{a}$ and $\vec{b}$, if a triangle ABC built on the vectors $\vec{a}$ and $\vec{b}$ that $\overrightarrow{CB}=\vec{a}$, $\overrightarrow{CA}=\vec{b}$:

a) $\vec{b}-\frac{1}{2}\vec{a}$; b) $\frac{1}{2}\vec{a}-\vec{b}$; c) $\vec{a}-\frac{1}{2}\vec{b}$; d) $\frac{1}{2}\vec{b}-\vec{a}$.

9. Radius-vector magnitude of the point A is 6. It forms an angle 45^0 with the x-axis and an angle 60^0 with the y-axis. Determine the coordinates of the point A if its z-coordinate is negative:

a) $(-3\sqrt{2},3,-3)$; b) $(3\sqrt{2},-3,-3)$;

c) $(3\sqrt{2},3,-3)$; d) $(-3\sqrt{2},3,-3)$.

10. Vector $\vec{a}$ forms the same angles 60^0 with the x-axis and with the z-axis. Determine the angle between the vector $\vec{a}$ and the y-axis

a) 45^0 or 135^0; b) -30^0 or 150^0;

c) -45^0 or 135^0; d) 30^0 or 150^0.

11. What condition must be satisfied for the vectors $\vec{a}$ and $\vec{b}$ if vectors $\vec{a}-\vec{b}$ and $\vec{a}+\vec{b}$ are collinear:

a) $\vec{a}\perp\vec{b}$; b) $\vec{a}//\vec{b}$; c) $\vec{a}=\vec{b}$; d) $\vec{a}=-\vec{b}$.

12. Does this equality $\left|\vec{a}\times\vec{b}\right|^2+\left(\vec{a}\cdot\vec{b}\right)^2=\left|\vec{a}\right|^2\cdot\left|\vec{b}\right|^2$ true

a) no; b) yes, only if $\vec{a}//\vec{b}$; c) yes; d) yes, only if $\vec{a}\perp\vec{b}$.

13. Find $\left|\vec{a}\times\vec{a}\right|$

a) $\left|\vec{a}\right|^2$; b) 0; c) $\left|\vec{a}\right|$; d) $2\left|\vec{a}\right|$.

14. What is a feature vectors $\vec{a},\vec{b}$ have if the following relation holds for them: $\dfrac{1}{\left|\vec{a}\right|}\vec{a}=\dfrac{1}{\left|\vec{b}\right|}\vec{b}$

a) $\vec{a}//\vec{b}$; b) vectors $\vec{a},\vec{b}$ have the same direction;

c) $\vec{a}\perp\vec{b}$; d) $\vec{a}=\vec{b}$.

15. What condition must the vectors $\vec{a},\vec{b}$ satisfy so that the vectors $3\vec{a}+\vec{b}$ and $\vec{a}-3\vec{b}$ were collinear

a) $\vec{a}=-\vec{b}$; b) $\vec{a}\perp\vec{b}$; c) $\vec{a}=\vec{b}$; d) $\vec{a}//\vec{b}$.

16. Knowing the coordinates of the vertex $A(1,-6,-3)$ and sides $\overrightarrow{AB}=(0,3,5)$ and $\overrightarrow{BC}=(4,2,-1)$, find the length of the edge:

a) $7{,}8$; b) $6{,}3$; c) $7{,}5$; d) $8{,}2$.

17. Do it $\vec{a}\cdot\left(\vec{a}+2\vec{b}\right)+\left(\vec{a}+\vec{b}\right)^2$ if $\left|\vec{a}\right|=4$, $\left|\vec{b}\right|=5$, and angel $\angle\left(\vec{a},\vec{b}\right)=60^0$:

a) 97; b) 137; c) $63{,}8$; d) $68{,}5$.

18. Find the projection of the vector $\vec{a}=(1,-3,1)$ on the vector $\overrightarrow{PQ}$ if points $P(-5,7,-5)$ and $Q(7,-9,9)$ are given:

a) 5; b) 3; c) -3; d) 7.

19. Three vertices $A(1,-1,2)$, $B(5,-6,2)$, $C(1,3,-1)$ of a triangle ABC are given. Calculate the length of the height drawn from the vertex B to the edge AC:

a) $2{,}5$; b) 5; c) $12{,}5$; d) 25.

20. The coordinates of the triangle vertices $O(0,0)$, $A(2a,0)$, $B(a,-a)$ are given. Find the angle formed by the edge OB and median OM of this triangle

a) $\varphi=\arccos\frac{\sqrt{2}}{2}$; b) $\varphi=\arccos\frac{\sqrt{5}}{2}$;

c) $\varphi=\arccos\frac{1}{\sqrt{5}}$; d) $\varphi=\arccos\frac{2}{\sqrt{5}}$.

21. Calculate a work of a force $\vec{F}=(3,-2,-5)$ applied to the point $A(2,-3,5)$ moving it to the point $B(3,-2,-1)$:

a) 35; b) 29; c) 31; d) 33.

22. Which of the following equalities is true:

a) $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\left(\vec{b}\times\vec{a}\right)\cdot\vec{c}$; b) $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\left(\vec{c}\times\vec{b}\right)\cdot\vec{a}$;

c) $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\left(\vec{a}\times\vec{c}\right)\cdot\vec{b}$; d) $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\left(\vec{b}\times\vec{c}\right)\cdot\vec{a}$.

23. Calculate the volume of the tetrahedron and the length of its height dropped from the vertex D if its vertices are $A(2,3,2)$, $B(4,1,-2)$, $C(6,3,7)$, $D(-5,-4,-8)$:

a) $V=\frac{154}{3}$, $h=\frac{11}{2}$; b) $V=308$, $h=66$;

c) $V=308$, $h=11$; d) $V=\frac{154}{3}$, $h=11$.

24. The vectors $\vec{a}=-\vec{i}+\vec{j}$, $\vec{b}=-3\vec{j}+2\vec{k}$, $\vec{c}=-\vec{5i}-3\vec{j}-\vec{k}$ are given. Find a vector $\vec{x}$ satisfying the following conditions: $\vec{a}\cdot\vec{x}=38$, $\vec{b}\cdot\vec{x}=133$, $\vec{c}\cdot\vec{x}=0$.

a) $\vec{x}=-\overrightarrow{25i}+13\vec{j}+86\vec{k}$; b) $\vec{x}=-\overrightarrow{25i}+13\vec{j}+65\vec{k}$;

c) $\vec{x}=\overrightarrow{25i}-13\vec{j}-86\vec{k}$; d) $\vec{x}=\overrightarrow{13i}-25\vec{j}+86\vec{k}$.

25. Calculate a moment of a force $\vec{F}=(1,-5,6)$ relative to the point $B(1,3,-4)$ if it applied to the point $A(4,2,1)$

a) $M_B=(19,-13,-14)$; b) $M_B=(31,23,-61)$;

c) $M_B=(-19,13,14)$; d) $M_B=(-31,-23,16)$.

26. Simplify the expression $3\vec{i}\left(\vec{j}\times\vec{k}\right)+5\vec{j}\left(\vec{i}\times\vec{k}\right)-6\vec{k}\left(\vec{i}\times\vec{j}\right)$

a) -14; b) 2; c) -8; d) 8.

25. The scalar triple product of non-zero vectors is zero: $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=0$. What can we say about the relative position of the vectors

a) vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular;

b) $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=0$ does not make sense;

c) vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar;

d) vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanar.

27. Vectors $\vec{a}$, $\vec{b}$, $\vec{c}$,..., $\vec{r}$ are linearly dependent if there are such coefficients λ_1, λ_2, …, λ_n then

a) $\lambda_1\vec{a}+\lambda_2\vec{b}+\lambda_3\vec{c}+...+\lambda_n\vec{r}=0$ and $\lambda_1^2+\lambda_2^2+...+\lambda_n^2=0$;

b) $\lambda_1\vec{a}+\lambda_2\vec{b}+\lambda_3\vec{c}+...+\lambda_n\vec{r}\neq 0$ and $\lambda_1^2+\lambda_2^2+...+\lambda_n^2=0$;

c) $\lambda_1\vec{a}+\lambda_2\vec{b}+\lambda_3\vec{c}+...+\lambda_n\vec{r}\neq 0$ and $\lambda_1^2+\lambda_2^2+...+\lambda_n^2\neq 0$;

d) $\lambda_1\vec{a}+\lambda_2\vec{b}+\lambda_3\vec{c}+...+\lambda_n\vec{r}=0$ and $\lambda_1^2+\lambda_2^2+...+\lambda_n^2\neq 0$.

28. Decomposition of a vector $\vec{a}$ in the basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is given $\vec{a} = -2\vec{e}_1 + 3\vec{e}_2 - \vec{e}_3$. Write vector components

a) $-2\vec{e}_1, 3\vec{e}_2, -\vec{e}_3$; b) $2\vec{e}_1, 3\vec{e}_2, \vec{e}_3$;

c) $-2\vec{e}_1, 3\vec{e}_2, \vec{e}_3$; d) $-2, 3, -1$.

29. Vectors $\vec{a} = (3,-1)$, $\vec{b} = (3,-1)$, $\vec{c} = (3,-1)$ are given on the plane. Decompose vector $\vec{p} = \vec{a} + \vec{b} + \vec{c}$ into a vectors basis $\vec{a}$ and $\vec{b}$

a) $\vec{p} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}$; b) $\vec{p} = 2\vec{a} - 3\vec{b}$;

c) $\vec{p} = 2\vec{a} + 5\vec{b}$; d) $\vec{p} = \vec{a} - 2\vec{b}$.

30. What condition must the vectors $\vec{a}, \vec{b}$ satisfy so that the vector $\vec{s} = \vec{a} + \vec{b}$ divided in half the angle between the vectors $\vec{a}$ and $\vec{b}$

a) $\vec{a} = \vec{b}$; b) $\vec{a} \perp \vec{b}$; c) $\angle(\vec{a}, \vec{b}) < \frac{\pi}{2}$; d) $|\vec{a}| = |\vec{b}|$.

31. Find the unit vector that is perpendicular to the vectors $\vec{a} = \vec{i} + \vec{j} + 2\vec{k}$ and $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$

a) $\frac{1}{\sqrt{11}}(\vec{i} - 3\vec{j} + \vec{k})$ or $\frac{1}{\sqrt{11}}(-\vec{i} + 3\vec{j} - \vec{k})$;

b) $\frac{1}{\sqrt{2}}(\vec{i} - 3\vec{j} + \vec{k})$ or $\frac{1}{\sqrt{11}}(-\vec{i} + 3\vec{j} + \vec{k})$;

c) $\frac{1}{\sqrt{7}}(\vec{i} - 3\vec{j} + \vec{k})$ or $\frac{1}{\sqrt{7}}(-\vec{i} + 3\vec{j} - \vec{k})$;

d) $\vec{i} - 3\vec{j} + \vec{k}$ or $-\vec{i} + 3\vec{j} - \vec{k}$.

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ALPHABETICAL INDEX

APPENDIX A

The standard basis vectors (orts)

The unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ are *standard basis vectors* (in the other words, orts). They are located at the coordinate axes and shown the positive direction. They are similar to unit segments on the coordinate axes in a Cartesian coordinate system (Figure A.1).

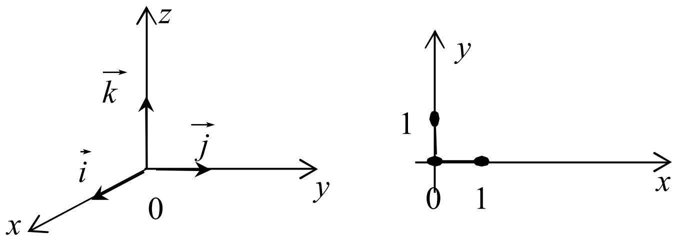

Figure A.1

In the three dimensional space there are three standard basis vectors,

$$\vec{i} = (1,0,0);\ \vec{j} = (0,1,0);\ \vec{k} = (0,0,1).$$

In the two dimensional space there are two standard basis vectors,

$$\vec{i} = (1,0) \text{ and } \vec{j} = (0,1).$$

Note that standard basis vectors are also unit vectors.

APPENDIX B

Force and reaction

Force is expressed in pressure and in tension. The most common efforts are those created by gravity of the Earth, for examples, a book presses on a table; suspended load stretches the suspension. As it is known in mechanics, there are free and non-free bodies. Bodies restricting the movement freedom of a given body are called *links*. And the forces with which links act on the body are called *link reactions*. One of the main principles of mechanics is the principle of exemption from links, according to which a non-free body can be considered as a free body if we discard the links acting on it and replace them with forces, i.e., link reactions. The direction of the reactions depends on the direction of the links and the loading scheme. The reaction of a “weightless” cable (thread, chain, rod, bar) is always directed along the cable (thread, chain, bar) (figure B.1).

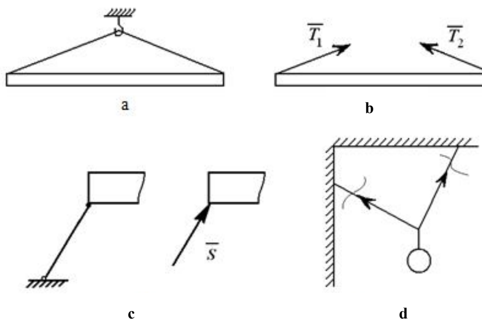

Figure B.1

a – the beam hangs on two cables; б – the action of the cables is replaced by forces T_1 и T_2; c – ideal bar link; d – perfect thread link

APPENDIX C

Right-hand and left-hand coordinate system

Let vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ be three nonzero vectors which are not collinear to the one plane taken in a specified order and they have the common starting point, and they are not coplanar vectors.

The system of three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is the *right-hand coordinate system* (Figure C.1) if the shortest rotation of the vector $\vec{a}$ to the vector $\vec{b}$ is performed counterclockwise for the observer which is at the end of the vector $\vec{c}$. If this rotation is performed clockwise, then such a system of vectors is the *left-hand coordinate system* (Figure C.2).

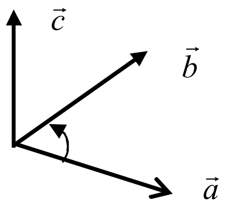

Figure C.1

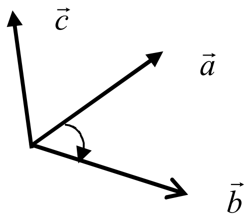

Figure C.2

Note, that the system of three vectors parallel to one plane is “neutral”: it is neither right nor left.

If the vectors system $\vec{a}$, $\vec{b}$, $\vec{c}$ is left-hand, then the vectors system $\vec{b}$, $\vec{a}$, $\vec{c}$ is right-hand. So, two systems

$$\vec{a}, \vec{b}, \vec{c} \text{ and } \vec{b}, \vec{a}, \vec{c}$$

have an opposite orientation.

APPENDIX D

Vector moment of a force

The vector moment of a force with respect to a point is the vector applied at this point and equaled in magnitude (Figure D.1) the product of the force on the shoulder of the force with respect to this point.

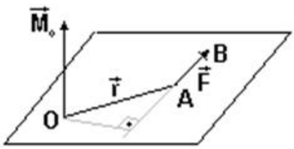

Figure D.1

The vector moment is a perpendicular to the plane in which the force and the point lie, so that from its end you can see the force “tendency” to rotate the body counterclockwise. Then,

$$\overrightarrow{M} = \overrightarrow{OA} \times \overrightarrow{F} ,$$

$$\left|\overrightarrow{M}\right| = \left|\overrightarrow{OA} \times \overrightarrow{F}\right| = \left|\overrightarrow{OA}\right| \cdot \left|\overrightarrow{F}\right| \cdot \sin \angle\left(\overrightarrow{OA}, \overrightarrow{F}\right) = \left|\vec{r}\right| \cdot \left|\overrightarrow{F}\right| \cdot \sin \angle\left(\vec{r}, \overrightarrow{F}\right).$$

The vector moment of a force relative to a point does not change when the force glides along the line of action and is equal to zero if the line passes through the point O.

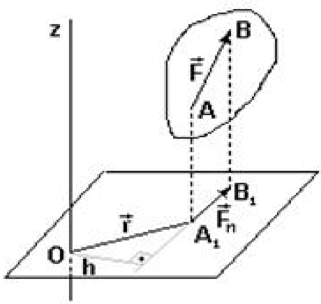

Figure D.2

The moment of a force with respect to the axis is the algebraic moment of the projection of this force onto a plane is perpendicular to the axis (Figure D.2), relative to the intersection point of the axis with this plane.

Note, that the moment of the force with respect to the axis is zero if the force and axis lie in the same plane.

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