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Посібник містить основні відомості про матриці, визначники, методи розв'язання систем лінійних алгебраїчних рівнянь, питання для самоконтролю, приклади завдань прикладного спрямування, у процесі вирішення яких використовуються розглянуті методи.

Посібник призначений для студентів, викладачів і читачів, які цікавляться питаннями лінійної алгебри

Sytnykova Yu. V.

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The tutorial contains the basic information about matrices, determinants, methods for solving systems of linear algebraic equations, questions for self-control, examples of problems of an applied nature in the process of solving which the methods considered are used.

The manual is intended for students, teachers and readers who are interested in questions of linear algebra

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CONTENT

PREFACE

The basis of the tutorial is a series of lectures on advanced mathematics taught at all the faculties of O. M. Beketov National University of Urban Economy in Kharkiv.

The purpose of the tutorial is the practical application of the mathematical apparatus to solve problems from the section "Linear algebra". It contains 12 paragraphs, each of which provides the necessary theoretical information on the definitions and basic mathematical concepts of the paragraph. The most complicated theoretical issues are accompanied by additional explanations of the concepts (without proof) with supplementary exercises and tasks. A feature of the manual is the presence of a large number of applied tasks which help to explain the applied nature of the mathematical apparatus of this topic.

The tutorial includes typical tasks and suggests the methods of their solution in detail. At the end of each paragraph the tasks for independent work are given to consolidate the material studied. At the end of the tutorial there are the answers. The appendices present separate definitions, theorems and problems, the solution of which causes difficulties, or is an extra task to solve more complex problems.

At the end of the book, an alphabetical index is submitted. It may be useful for a quick search for reference information on a mathematical term, as well as the answers to the required questions.

The authors hope that the presentation of the text in English will allow replenishing the vocabulary of the English language of both students and teachers, and will improve the quality of foreign students' training.

Acknowledgement

The authors express their gratitude to Svitlana Zubenko, an Associate Professor of the department of Foreign Languages of O. M. Beketov National University of Urban Economy in Kharkiv for the assistance rendered in the process of preparing this tutorial, for her valuable linguistic remarks and constructive advice in editing the text in English.

1 The concept of the determinant. Rules of calculation and properties of determinants

Determinant is defined as the number written as a square table. The element of determinant is a number denoted as a_{ij}, where indices i and j indicate the location of this element in the table of numbers. The index i is a number of a row, where the element a_{ij} locates, the index j is a number of a column. The determinant is recorded as a table of numbers in direct parentheses. and is marked Δ_n, where index n indicates the order of the determinant. *The order of the determinant* is considered the number of its rows (columns). For example, $\begin{vmatrix} -1 & 3 \\ 7 & 0 \end{vmatrix}$ is the determinant of second order, because there are two rows and two columns, so it can be marked as $\Delta_2 = \begin{vmatrix} -1 & 3 \\ 7 & 0 \end{vmatrix}$. If we say that the number 3 is the element of a determinant Δ_2, which is located at the first row and the second column, we can denote it as: $a_{12} = 3$. $\Delta_3 = \begin{vmatrix} 3 & -2 & 6 \\ 5 & 1 & 0 \\ -2 & -7 & 3 \end{vmatrix}$ is the determinant of the third order. By the way, the number of elements of any determinant is equal n^2. This can be verified by paying attention to the previous examples: the amount of elements in the determinant of the second order is four, and in the determinant of the third order there are nine of them.

The second-order determinant is calculated by the "cross" rule (the scheme of it is shown at the figure 1.1):

$$\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12} .$$

The third-order determinant is calculated by «asterisk rule» («rule of triangles»; Sarrus' rule or Sarrus' scheme is named after the French mathematician Pierre Frédéric Sarrus.) (the scheme is shown at the figure 1.2):

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\cdot a_{22}\cdot a_{33} + a_{12}\cdot a_{23}\cdot a_{31} + a_{21}\cdot a_{32}\cdot a_{13} -$$

$$- a_{13}\cdot a_{22}\cdot a_{31} - a_{12}\cdot a_{21}\cdot a_{33} - a_{32}\cdot a_{23}\cdot a_{11}.$$

The rules can be represented schematically and specified as follows:

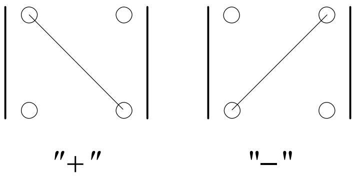


Figure 1.1

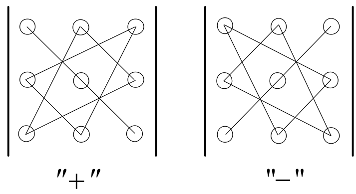


Figure 1.2

Example 1.1 Calculate $\begin{vmatrix} 2 & 3 & -2 \\ -1 & -2 & 3 \\ 5 & 4 & -2 \end{vmatrix}$.

Solution: $\Delta_3 = \begin{vmatrix} 2 & 3 & -2 \\ -1 & -2 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 2\cdot(-2)\cdot(-2) + 3\cdot 3\cdot 5 +$

$$+(-1)\cdot 4\cdot(-2) - 5\cdot(-2)\cdot(-2) - 3\cdot(-1)\cdot(-2) - 3\cdot 4\cdot 2 =$$

$$= -8 + 45 + 8 - 20 - 6 - 24 = -5.$$

Answer: -5.

The determinant of an arbitrary order n looks like this:

$$\Delta_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}.$$

The main diagonal of the determinant is the diagonal, which consists of elements $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$. Another diagonal is called a secondary diagonal.

Consider the question of calculating a determinant of arbitrary order. First of all, we should find out the following concepts, as a minor and a cofactor of the element.

Minor M_{ij} of the element a_{ij} is the determinant of the $(n-1)$ order that is obtained by deleting i row and j column, at the intersection of which this element is located. For example, find the minor M_{23} in the determinant $\Delta_3 = \begin{vmatrix} 3 & -2 & 6 \\ 5 & 1 & 0 \\ -2 & -7 & 3 \end{vmatrix}$. According to the definition we need to delete the second row and the third column, so we get the determinant as: $M_{23} = \begin{vmatrix} 3 & -2 \\ -2 & -7 \end{vmatrix}$.

Note 1.1 In the determinant of the second order $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, for instant, the minor of the element a_{11} is the element a_{22}, which can be considered as the first-order determinant, and which is obtained by deleting the first row and the first column.

The cofactor A_{ij} *of the element* a_{ij} is:

$$A_{ij} = (-1)^{i+j} \cdot M_{ij} . \tag{1.1}$$

This is a minor, whose sign depends on the parity or oddness of the sum of the indices of this element. For example, find the cofactors A_{22} and A_{32} in the given determinant $\Delta_3 = \begin{vmatrix} 0 & -7 & -1 \\ 8 & -5 & 0 \\ -2 & 7 & 4 \end{vmatrix}$. For this we will use the formula (1.1) and get:

$$A_{22} = (-1)^{2+2} \cdot M_{22} = 1 \cdot \begin{vmatrix} 0 & -1 \\ -2 & 4 \end{vmatrix} = 0 \cdot 4 - (-1) \cdot (-2) = -2 ,$$

$$A_{32} = (-1)^{3+2} \cdot M_{32} = (-1) \cdot \begin{vmatrix} 0 & -1 \\ 8 & 0 \end{vmatrix} = -(0 \cdot 0 - 8 \cdot (-1)) = -8 .$$

The calculations of determinants of any order are performed by their decomposition by the elements of the row (or column) (the Laplace expansion, named after Pierre-Simon Laplace, the so-called *cofactor expansion*), i.e.: the determinant is equal to the sum of products of elements of a certain row (column) by their cofactor. It is:

$$\Delta = \sum_{k=1}^{n} a_{ik} \cdot A_{ik} \text{ or } \Delta = \sum_{k=1}^{n} a_{kj} \cdot A_{kj} . \tag{1.2}$$

Example 1.2 Calculate the determinant $\begin{vmatrix} 2 & 1 & 5 & 1 \\ 3 & 2 & 1 & 2 \\ 1 & 2 & 3 & -4 \\ 1 & 1 & 5 & 1 \end{vmatrix}$.

Solution. Calculate the fourth order determinant, decomposing it by the elements of the first column. According to the formula (1.2) we will have the following decomposition:

$$\begin{vmatrix} 2 & 1 & 5 & 1 \\ 3 & 2 & 1 & 2 \\ 1 & 2 & 3 & -4 \\ 1 & 1 & 5 & 1 \end{vmatrix} = 2 \cdot A_{11} + 3 \cdot A_{21} + 1 \cdot A_{31} + 1 \cdot A_{41} =$$

$$= 2 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & -4 \\ 1 & 5 & 1 \end{vmatrix} + 3 \cdot (-1)^{2+1} \cdot \begin{vmatrix} 1 & 5 & 1 \\ 2 & 3 & -4 \\ 1 & 5 & 1 \end{vmatrix} +$$

$$+ 1 \cdot (-1)^{3+1} \cdot \begin{vmatrix} 1 & 5 & 1 \\ 2 & 1 & 2 \\ 1 & 5 & 1 \end{vmatrix} + 1 \cdot (-1)^{4+1} \cdot \begin{vmatrix} 1 & 5 & 1 \\ 2 & 1 & 2 \\ 2 & 3 & -4 \end{vmatrix} =$$

$$= 2 \cdot (6 + 20 - 4 - 6 - 2 + 40) + 3 \cdot (-1) \cdot 0 + 1 \cdot 0 +$$

$$+ 1 \cdot (-4 + 20 + 6 - 2 - 6 + 40) = 108 + 54 = 162 .$$

Answer: 162.

Properties of determinants:

1. The value of the determinant will not change if its rows are replaced by the corresponding columns, and the columns are replaced by the corresponding rows:

$$\begin{vmatrix} -9 & 8 \\ 1 & 5 \end{vmatrix} = -9 \cdot 5 - 8 \cdot 1 = -45 - 8 = -53 ,$$

$$\begin{vmatrix} -9 & 1 \\ 8 & 5 \end{vmatrix} = -9 \cdot 5 - 1 \cdot 8 = -45 - 8 = -53 .$$

2. The determinant will be zero if it contains a row (column) with zero elements.

3. If we swap the two rows (columns), then the sign of the determinant will switch.

$$\begin{vmatrix} 0 & -1 & 4 \\ 2 & 9 & -5 \\ -4 & 3 & 5 \end{vmatrix} = -\begin{vmatrix} 2 & 9 & -5 \\ 0 & -1 & 4 \\ -4 & 3 & 5 \end{vmatrix}.$$

4. The determinant will be zero if it contains two identical or proportional rows (columns):

$$\begin{vmatrix} 5 & 1 & -7 \\ -15 & -3 & 21 \\ -2 & 3 & 5 \end{vmatrix} = 0.$$

5. A factor, common to elements of a certain row (column), can be extracted before sign of a determinant:

$$\begin{vmatrix} 10 & -1 & 4 \\ 8 & 1 & -7 \\ -2 & 3 & 5 \end{vmatrix} = 2 \cdot \begin{vmatrix} 5 & -1 & 4 \\ 4 & 1 & -7 \\ -1 & 3 & 5 \end{vmatrix}.$$

6. If the elements of a certain row (column) of a determinant are represented as the sum of two terms, then this determinant will be equal to the sum of two determinants:

$$\Delta_3 = \begin{vmatrix} 12 & -2 & -1 \\ 5 & 11 & -7 \\ -2 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 5+7 & -2 & -1 \\ 3+2 & 11 & -7 \\ -3+1 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 5 & -2 & -1 \\ 3 & 11 & -7 \\ -3 & 3 & 3 \end{vmatrix} + \begin{vmatrix} 7 & -2 & -1 \\ 2 & 11 & -7 \\ 1 & 3 & 3 \end{vmatrix}.$$

7. The value of the determinant does not change if the elements of a certain row (column), multiplied by the same

number that is not equal to zero, add to elements of a different row (column).

Check out this property on an example calculating the determinant $\begin{vmatrix} 2 & -1 & 4 \\ 5 & 1 & -7 \\ -2 & 3 & 5 \end{vmatrix}$.

For this we do the following:

1) add the first row to the second row multiplied by two, and calculate the determinant by the rule of triangles:

$$\begin{vmatrix} 2 & -1 & 4 \\ 5 & 1 & -7 \\ -2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 2+5\cdot 2 & -1+1\cdot 2 & 4+(-7)\cdot 2 \\ 5 & 1 & -7 \\ -2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 12 & 1 & -10 \\ 5 & 1 & -7 \\ -2 & 3 & 5 \end{vmatrix} =$$

$$= 60+14-150-20-25+252 = 131;$$

2) calculate the determinant by the rule of triangles:

$$\begin{vmatrix} 2 & -1 & 4 \\ 5 & 1 & -7 \\ -2 & 3 & 5 \end{vmatrix} = 10-14+60+8+25+42 = 131.$$

As you can see, the value of a given determinant really has not changed even after our operations using the above mentioned property number seven.

Note 1.2 Elements of the determinant may be not only numbers, but other objects as well.

Example 1.3 Solve the equation $\begin{vmatrix} x & -2 & 2 \\ 1 & x & -1 \\ 1 & -1 & 1 \end{vmatrix} = 0$.

Solution. We will use the rule of triangles:

$$\begin{vmatrix} x & -2 & 2 \\ 1 & x & -1 \\ 1 & -1 & 1 \end{vmatrix} = x^2+2-2-2x+2-x = x^2-3x+2,$$

$$x^2-3x+2=0,$$
$$D=9-4\cdot 2=1,$$
$$x_{1,2}=\frac{3\pm 1}{2},\ x_1=2,\ x_2=1.$$

Answer: $x_1=2$, $x_2=1$.

Example 1.4 Solve the equation $\begin{vmatrix} \sin\frac{x}{2} & \cos\frac{x}{2} \\ \cos\frac{x}{2} & \sin\frac{x}{2} \end{vmatrix} = 1$.

Solution. According to the rule of triangles we get:

$$\begin{vmatrix} \sin\frac{x}{2} & \cos\frac{x}{2} \\ \cos\frac{x}{2} & \sin\frac{x}{2} \end{vmatrix} = \sin\frac{x}{2}\cdot\sin\frac{x}{2}-\cos\frac{x}{2}\cdot\cos\frac{x}{2}=\sin^2\frac{x}{2}-\cos^2\frac{x}{2}=-\cos x,$$

$$\begin{vmatrix} \sin\frac{x}{2} & \cos\frac{x}{2} \\ \cos\frac{x}{2} & \sin\frac{x}{2} \end{vmatrix} = 1,\ -\cos x=1,\ \cos x=-1,$$
$$x=\arccos(-1)+2\pi n,\ x=\pi+2\pi n,\ n\in Z.$$

Answer: $x=\pi+2\pi n$, $n\in Z$

Questions for self-control

1. What is a determinant?

2. What is a minor?

3. What is a cofactor of a determinant element?

4. By what rule the value of the determinant of the n-th order is calculated?

5. Formulate the rules of the "cross" and "triangles" for calculating respectively the determinants of the second and third order

6. What are the basic properties of the determinant? Which of them can we use to calculate it.

7. How a determinant of a triangular form is calculated?

8. Will be the value of the determinant changed if the elements of some column are multiplied by 5? If so, how much?

9. Which of the properties can we use to simplify calculation the determinant of any order? Explain your answer

Tasks for revision

1.1 Calculate the determinant $\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 2 & a+3 & b+4 \\ 2 & c+3 & d+4 \end{vmatrix}$ by the most convenient way.

1.2 Calculate the determinant $\Delta = \begin{vmatrix} 10 & 1 & 0 & 0 & 0 \\ 11 & 10 & 1 & 0 & 0 \\ 0 & 11 & 10 & 1 & 0 \\ 0 & 0 & 11 & 10 & 1 \\ 0 & 0 & 0 & 11 & 10 \end{vmatrix}$ by the most convenient way.

1.3 Calculate the given determinant by the most convenient way

$$\Delta = \begin{vmatrix} 1+b & 1 & 1 & 1 \\ 1 & 1-b & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1-b \end{vmatrix}.$$

2 Matrices. Operations with matrices. The concept of an inverse matrix and algorithm used to find it

A matrix is defined as an ordered rectangular array (table) of numbers. If a matrix has m rows and n columns, we will say that it has dimension as $m \times n$. If a matrix has only one column and m rows, we name it a matrix-column; if a matrix has only one row and n columns we name it a matrix-row. Matrix with an equal number of rows and columns, is called a *square matrix*. *A symmetric matrix* is a square matrix, in which its elements are located symmetrically the main diagonal are equal to each others, that is: $a_{kl} = a_{lk}$. *A diagonal matrix* is called a square matrix in which all elements, except those which are on the main diagonal, equal zero.

For examples:

Symmetric	Diagonal	Upper Triangular	Lower Triangular
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & -5 \\ 3 & -5 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 7 & 0 \\ 3 & -5 & 4 \end{pmatrix}$

Identity matrix is a diagonal matrix, the diagonal elements of which are units, and denote by letter E:

$$E = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Matrices A and B are *equal*, if they have the equal dimension $m \times n$ and their corresponding elements are equal

$$A_{m\times n} = B_{m\times n} \quad \Leftrightarrow \quad a_{ij} = b_{ij}, \; i = \overline{1,m}; \; j = \overline{1,n} \; .$$

Operations with matrices:

1. The multiplication of matrix by the scalar. To do this, each element of the matrix must be multiplied by a given number.

Example 2.1 Find the matrix $C = 4A$, if $A = \begin{pmatrix} 3 & -4 & 2 \\ -8 & 1 & 5 \end{pmatrix}$.

Solution: according to the above mentioned rule we should multiply all elements of given matrix A by four

$$C = 4A = 4 \cdot \begin{pmatrix} 3 & -4 & 2 \\ -8 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 3\cdot 4 & -4\cdot 4 & 2\cdot 4 \\ -8\cdot 4 & 1\cdot 4 & 5\cdot 4 \end{pmatrix} = \begin{pmatrix} 12 & -16 & 8 \\ -32 & 4 & 20 \end{pmatrix}$$

.

Answer: $\begin{pmatrix} 12 & -16 & 8 \\ -32 & 4 & 20 \end{pmatrix}$.

2. Addition (or subtraction) of matrices. Two matrices A and B can be added or subtracted if and only if their dimensions are the same (i.e. both matrices have the same number of rows and columns). If matrices A and B have the

same dimension then the sum of them is found by adding the corresponding elements $a_{ij}+b_{ij}$.

Example 2.2 Find the sum of matrices A i B, if

$$A=\begin{pmatrix} 3 & -4 & 2 \\ -8 & 1 & 5 \end{pmatrix},\ B=\begin{pmatrix} -7 & 5 & -4 \\ -8 & 3 & 2 \end{pmatrix}.$$

Solution. Carry out the operation of addition between corresponding elements

$$A+B=\begin{pmatrix} 3 & -4 & 2 \\ -8 & 1 & 5 \end{pmatrix}+\begin{pmatrix} -7 & 5 & -4 \\ -8 & 3 & 2 \end{pmatrix}=\begin{pmatrix} 3-7 & -4+5 & 2-4 \\ -8-8 & 1+3 & 5+2 \end{pmatrix}=$$

$$=\begin{pmatrix} -4 & 1 & -2 \\ -16 & 4 & 7 \end{pmatrix}.$$

Answer: $\begin{pmatrix} -4 & 1 & -2 \\ -16 & 4 & 7 \end{pmatrix}$.

We emphasize that the operation of addition is commutative, i.e.

$$A+B=B+A.$$

3. Matrix multiplication. When the number of columns of the first matrix is the same as the number of rows in the second matrix then matrix multiplication can be performed. Matrix-result has the number of rows as the first matrix and the number of columns as the second matrix, i.e.: $A_{m\times k}\cdot B_{k\times n}=D_{m\times n}$. Elements of this matrix-result are obtained as the sum of the products of the row elements of the first matrix by the corresponding column elements of the second matrix.

Example 2.3 Find the product of matrices : $A\cdot B$ and $B\cdot A$:

a) $A=\begin{pmatrix}3 & -1 & 1\\ 2 & -2 & 3\end{pmatrix}$, $B=\begin{pmatrix}7 & 5\\ 0 & -8\end{pmatrix}$; b) $A=\begin{pmatrix}3 & -4\\ 2 & -1\end{pmatrix}$, $B=\begin{pmatrix}7 & 5\\ 0 & -8\end{pmatrix}$.

Solution: a) $A\cdot B=\begin{pmatrix}3 & -1 & 1\\ 2 & -2 & 3\end{pmatrix}\cdot\begin{pmatrix}7 & 5\\ 0 & -8\end{pmatrix}$ – unfortunately, these matrices cannot be multiplied, because the column number of the first matrix is three, and the row number of the second matrix is two, hence those matrices are not satisfied the above mentioned conditions. However, the multiplication $B\cdot A$ exists, find it:

$$B\cdot A=\begin{pmatrix}7 & 5\\ 0 & -8\end{pmatrix}\cdot\begin{pmatrix}3 & -1 & 1\\ 2 & -2 & 3\end{pmatrix}=$$

we will multiply the each element along the first row of the matrix A with the corresponding elements down the first column of the matrix B, and add the results, after this we will continue to do it with the all columns, changing the row, repeating the operations

$$=\begin{pmatrix}7\cdot 3+5\cdot 2 & -1\cdot 7+5\cdot(-2) & 7\cdot 1+5\cdot 3\\ 0\cdot 3+(-8)\cdot 2 & 0\cdot(-1)-8\cdot(-2) & 0\cdot 1+(-8)\cdot 3\end{pmatrix}=$$

$$=\begin{pmatrix}31 & -17 & 22\\ -16 & 16 & -24\end{pmatrix};$$

b) given matrices have the same dimension, they are square matrices, so multiplications as $A\cdot B$, as well as $B\cdot A$ exist. Find them.

$$A\cdot B=\begin{pmatrix}3 & -4\\ 2 & -1\end{pmatrix}\cdot\begin{pmatrix}7 & 5\\ 0 & -8\end{pmatrix}=\begin{pmatrix}3\cdot 7+(-4)\cdot 0 & 3\cdot 5+(-4)\cdot(-8)\\ 2\cdot 7+(-1)\cdot 0 & 2\cdot 5+(-1)\cdot(-8)\end{pmatrix}=$$

$$=\begin{pmatrix} 21 & 47 \\ 14 & 18 \end{pmatrix},$$

$$B\cdot A=\begin{pmatrix} 7 & 5 \\ 0 & -8 \end{pmatrix}\cdot\begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix}=\begin{pmatrix} 7\cdot 3+5\cdot 2 & 7\cdot(-4)+5\cdot(-1) \\ 0\cdot 3+(-8)\cdot 2 & (-4)\cdot 0+(-1)\cdot(-8) \end{pmatrix}=$$

$$=\begin{pmatrix} 31 & -33 \\ -16 & 8 \end{pmatrix}.$$

Answer: a) $A\cdot B$ does not exist, $B\cdot A=\begin{pmatrix} 31 & -17 & 22 \\ -16 & 16 & -24 \end{pmatrix}$,

b) $A\cdot B=\begin{pmatrix} 21 & 47 \\ 14 & 18 \end{pmatrix}$, $B\cdot A=\begin{pmatrix} 31 & -33 \\ -16 & 8 \end{pmatrix}$.

As we see, multiplication of matrices is not commutative, that is

$$A\cdot B\neq B\cdot A.$$

Note 2.1 Note some properties of the matrix product:

1) $AE=A$; $EA=A$; 2) $(AB)C=A(BC)$;

3) $A(B+C)=AB+AC$; 4) $\alpha(AB)=(\alpha A)B=A(\alpha B)$;

5) for square matrices A and B is correct the following

$$\det(AB)=\det(BA)=\det A\cdot\det B.$$

Note 2.2 For a square matrix, powers of a matrix A can be defined as,

$$A^2=A\cdot A,\ A^3=A\cdot A\cdot A=A\cdot A^2=A^2\cdot A.$$

Each square matrix A of the arbitrary order n is placed respectively the determinant, which can be written as

$$\det A = \Delta A = |A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

It is called the determinant of the matrix A.

Note 2.3 A determinant of a square matrix is defined in such a way that a scalar value is associated with the matrix that does not change with certain row or column operations on the matrix i.e., it is one of the scalar invariants of the matrix

Further, we need to know about the transposed matrix. The transpose matrix of a matrix A can be found by exchanging rows for columns. Matrix $A = (a_{ij})$ and the transpose of A^T is: $A^T = (a_{ji})$ where j is the column number and i is the row number of matrix A.

In the case of a square matrix (when $m = n$), the transposition can be used to check if a matrix is symmetric. For a symmetric matrix $A^T = A$.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}; \quad A^T = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

Example 2.4 Transpose the given matrix

$$A = \begin{pmatrix} -7 & 2 & -3 \\ 3 & -5 & 2 \end{pmatrix}.$$

Solution:
$$A^T = \begin{pmatrix} -7 & 3 \\ 2 & -5 \\ -3 & 2 \end{pmatrix}.$$

If the determinant of the matrix A is zero, i.e. $\det A = 0$, then such a matrix A is called *singular (degenerate) matrix*.

If the determinant of the matrix A is not zero, i.e.

$\det A \neq 0$, then the matrix A is called *non-singular (non-degenerate) matrix*.

A matrix A^{-1} is called *the inverse matrix* of the non-singular square matrix A, if the condition

$$A\,A^{-1} = A^{-1}\,A = E$$

is fulfilled.

Theorem. The unique inverse matrix A^{-1} exists for arbitrary non-singular square matrix A of the n-order, and it can be calculated by the formula

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix},$$

where A_{ij} are cofactors of all elements a_{ij} of the matrix A. (Without proof).

To find the inverse matrix, use the following *algorithm*:

1. Calculate the determinant of the matrix $\det A$. If it equals zero, then the inverse matrix does not exist.
2. Transpose the matrix and get A^T.
3. Calculate cofactors of all elements of the transposed matrix A^T and find the inverse matrix A^{-1}.
4. Check out the equality $A\,A^{-1} = A^{-1}\,A = E$.

Example 2.5 Find the inverse matrix to the matrix

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}.$$

Solution: Calculate the determinant of the matrix A:

$$\det A = \begin{vmatrix} -1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 1+0+2+1-2-0 = 2 \neq 0,$$

so the given matrix A is a non-singular matrix and we can find the inverse matrix A^{-1}.

Transpose the matrix A and obtain the matrix A^T:

$$A^T = \begin{pmatrix} -1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

Calculate cofactors of all elements of the transposed matrix A^T and write down the inverse matrix:

$$A_{11}^T = (-1)^{1+1} \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = -1, \quad A_{12}^T = (-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1,$$

$$A_{13}^T = (-1)^{3+1} \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 3, \quad A_{21}^T = (-1)^{2+1} \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0,$$

$$A_{22}^T = (-1)^{2+2} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2, \quad A_{23}^T = (-1)^{2+3} \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} = 4,$$

$$A_{31}^T = (-1)^{1+3} \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 1, \quad A_{32}^T = (-1)^{2+3} \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = 1,$$

$$A_{33}^T = (-1)^{3+3} \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = -1, \quad A^{-1} = \frac{1}{2} \begin{pmatrix} -1 & -1 & 3 \\ 0 & -2 & 4 \\ 1 & 1 & -1 \end{pmatrix}.$$

Check out:

$$A^{-1}\cdot A=\frac{1}{2}\begin{pmatrix}-1 & -1 & 3\\ 0 & -2 & 4\\ 1 & 1 & -1\end{pmatrix}\cdot\begin{pmatrix}-1 & 1 & 1\\ 2 & -1 & 2\\ 1 & 0 & 1\end{pmatrix}=$$

$$=\frac{1}{2}\begin{pmatrix}1-2+3 & -1+1+0 & -1-2+3\\ 0-4+4 & 0+2+0 & 0-4+4\\ -1+2-1 & 1-1+0 & 1+2-1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}=$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=E.$$

Thus, the inverse matrix has been found correctly.

Answer: $A^{-1}=\frac{1}{2}\begin{pmatrix}-1 & -1 & 3\\ 0 & -2 & 4\\ 1 & 1 & -1\end{pmatrix}.$

Questions for self-control

1. What is a matrix?
2. Which of matrix is called non-degenerate?
3. How are doing the operations of adding (subtracting) matrices and multiplying the matrix by the number?
4. What is the difference between multiplication of the matrix by the scalar and the multiplication of the determinant by the number?
5. How is operation of multiplication of the matrixes carried out? What are the properties of this operation?
6. Which of matrix has determinant?
7. What is an inverse matrix and how is it calculated?
8. Does any matrix have an inverse matrix? Why?
9. How can you check the accuracy of the found inverse

matrix?

Tasks for revision

2.1 Check out the fifth propriety of matrices multiplication for the given matrices $A=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B=\begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}$.

2.2 Find A^3, if $A=\begin{pmatrix} a & a \\ a & a \end{pmatrix}$.

2.3 Find the inverse matrix A^{-1} for the given matrix A

$$A=\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

3 Systems of linear algebraic equations. Solution of a system. Methods of finding the solution of a linear algebraic equations system

A system of linear algebraic equations (SLAE) is a set of equations with m equations and n unknowns x_j $(j=\overline{1,n})$, it looks like

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1; \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m, \end{cases}$$

where a_{ij} $(i=\overline{1,m},\ j=\overline{1,n})$ and b_i $(i=\overline{1,m})$ are *given numbers*: a_{ij} $(i=\overline{1,m},\ j=\overline{1,n})$ are *coefficients of the* *unknowns*; b_i ($i=\overline{1,m}$) are *the constant term*s (*right parts*).

A system in which the number of equations is equal to the number of unknowns n, is called *the square system* of the n order.

The determinant of the square system

$$\Delta_n = \det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix},$$

composed of coefficients of the unknowns, is called the *main determinant of the system (or the coefficients matrix)*.

A system, where all constant terms are zeros, $b_i = 0$ ($i=\overline{1,m}$), is a homogeneous system, but if at least one of the constant terms is not equal to zero, then a system will be named non-homogeneous.

A system can be named as *compatible* if it has one solution at least, and a system is *incompatible* (contradictory) if it has no solution.

A homogeneous system is always *compatible*, because it has at least one trivial (zero-solution) solution $x_j = 0$ $(j=\overline{1,n})$.

A compatible system is *defined* if it has the unique solution, otherwise this system will be named as *indefinite*.

In matrix form the system of equations above can be written as:

$$A \cdot X = B,$$

where X is a matrix-column of unknowns, it has dimension $n \times 1$; A is a the coefficients matrix of the system composed of coefficients of the unknowns and has dimension as $m \times n$; B is a matrix-column of the constant terms (right parts), which has dimension $m \times 1$; $\widetilde{A}$ is an augmented matrix of the system which has dimension $m \times (n+1)$:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_m \end{pmatrix}; \quad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}; \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix};$$

$$\tilde{A} = \left(A \middle| B \right) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right).$$

It is clear that the coefficients matrix of a square system will also be square, therefore for such a matrix we can try to find the inverse matrix. Consider this possibility.

To do this, we solve the matrix equation $A \cdot X = B$. As we know, if a matrix A is non-singular, we can find the inverse matrix, it is A^{-1}. We multiply both sides of the matrix equation by the inverse matrix A^{-1}, and have

$$A^{-1} \cdot AX = A^{-1} \cdot B.$$

Considering it $A A^{-1} = A^{-1} A = E$, we get

$$E \cdot X = A^{-1} \cdot B,$$

$$X = A^{-1} \cdot B. \tag{3.1}$$

This formula (3.1) is a formula for finding the solution of square system and this method is called as *the matrix method*. Thus, if we need to solve SLAE by the matrix method we should do the following acts: firstly, find the inverse matrix

A^{-1} for the coefficients matrix A (using the algorithm to find the inverse matrix presented at the section 2); secondly, multiply the obtained inverse matrix A^{-1} by the matrix-column B of the constant terms.

Example 3.1 Solve the system of linear algebraic equations by the matrix method

$$\begin{cases} 3x_1 - 2x_2 + x_3 = 1; \\ x_1 - x_2 + 2x_3 = -3; \\ 2x_1 - x_2 + 3x_3 = -4. \end{cases}$$

Solution. Write down the coefficients matrix and find the inverse matrix for it (look at the algorithm to find the inverse matrix):

$$A = \begin{pmatrix} 3 & -2 & 1 \\ 1 & -1 & 2 \\ 2 & -1 & 3 \end{pmatrix},$$

$$\det A = \begin{vmatrix} 3 & -2 & 1 \\ 1 & -1 & 2 \\ 2 & -1 & 3 \end{vmatrix} = -9 - 1 - 8 + 2 + 6 + 6 = -4 \neq 0,$$

$$A^T = \begin{pmatrix} 3 & 1 & 2 \\ -2 & -1 & -1 \\ 1 & 2 & 3 \end{pmatrix},$$

$$A_{11}^T = \begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix} = -3 + 2 = -1, \quad A_{12}^T = -\begin{vmatrix} -2 & -1 \\ 1 & 3 \end{vmatrix} = -(-6+1) = 5,$$

$$A_{13}^T = \begin{vmatrix} -2 & -1 \\ 1 & 2 \end{vmatrix} = -4 + 1 = -3, \quad A_{21}^T = -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -(3-4) = 1,$$

$$A_{22}^T=\begin{vmatrix}3 & 2\\ 1 & 3\end{vmatrix}=9-2=7,\quad A_{23}^T=-\begin{vmatrix}3 & 1\\ 1 & 2\end{vmatrix}=-(6-1)=-5,$$

$$A_{31}^T=\begin{vmatrix}1 & 2\\ -1 & -1\end{vmatrix}=-1+2=1,\quad A_{32}^T=-\begin{vmatrix}3 & 2\\ -2 & -1\end{vmatrix}=-(-3+4)=-1,$$

$$A_{33}^T=\begin{vmatrix}3 & 1\\ -2 & -1\end{vmatrix}=-3+2=-1,\quad A^{-1}=\frac{1}{-4}\begin{pmatrix}-1 & 5 & -3\\ 1 & 7 & -5\\ 1 & -1 & -1\end{pmatrix},$$

$$X=A^{-1}\cdot B=\frac{1}{-4}\begin{pmatrix}-1 & 5 & -3\\ 1 & 7 & -5\\ 1 & -1 & -1\end{pmatrix}\cdot\begin{pmatrix}1\\ -3\\ -4\end{pmatrix}=-\frac{1}{4}\begin{pmatrix}-1-15+12\\ 1-21+20\\ 1+3+4\end{pmatrix}=$$

$$=-\frac{1}{4}\begin{pmatrix}-4\\ 0\\ 8\end{pmatrix}=\begin{pmatrix}1\\ 0\\ -2\end{pmatrix},$$

$$x_1=1,\ x_2=0,\ x_3=-2.$$

Let’s check our results. Substitute the obtained values of unknowns x_1, x_2, x_3 in all equations of the system:

$$\begin{cases}3\cdot 1-2\cdot 0-2=3-0-2=1;\\ 1-0+2\cdot(-2)=1-0-4=-3;\\ 2\cdot 1-0+3\cdot(-2)=2-0-6=-4.\end{cases}$$

Since we have got the identities, we can say that our obtained values of unknowns x_1, x_2, x_3 are correct.

Answer: $x_1=1$, $x_2=0$, $x_3=-2$.

You can also use Cramer's Rule to find a solution to a square system, which sometime is called the determinant's method.

Theorem 3.1 *(Cramer's Rule)*. If a determinant of a square system does not equal zero, then the system has one unique solution which can be calculated by the formula

$$x_j = \frac{\Delta_j}{\Delta},\ j = \overline{1,n}, \tag{3.2}$$

where Δ is a main determinant of the system composed of the system equations coefficients a_{ij} before unknowns x_j; Δ_j is the auxiliary determinant obtained from the main determinant Δ by replacing j-column with a column of the constant terms ($j = \overline{1,n}$).

For the system of linear algebraic equations, which consists of three equations with three unknowns x_1, x_2, x_3, it looks like

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1; \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2; \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3. \end{cases}$$

the solution according to Cramer's rule is performed as follows:

firstly, we should calculate the main determinant of the system composed of the coefficients for variables

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix};$$

secondly, we should calculate auxiliary determinants each of which is obtained by successively replacing the columns of the determinant by a column of the constant terms (it is numbers the following in the equations of the system after the sign of equal). Doing this we obtain three determinants:

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \ \Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \ \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix};$$

thirdly, to find the unknown, we need to use formulas, which can be received from the formula (3.2):

$$x_1 = \frac{\Delta_1}{\Delta}; \ x_2 = \frac{\Delta_2}{\Delta}; \ x_3 = \frac{\Delta_3}{\Delta}. \tag{3.3}$$

Example 3.2 Solve the square system by Cramer's rule

$$\begin{cases} 5x_1 - 3x_2 + x_3 = 2; \\ 3x_1 + x_2 - 5x_3 = 4; \\ x_1 - 2x_2 + 7x_3 = 3. \end{cases}$$

Solution. Calculate the system main determinant:

$$\Delta = \begin{vmatrix} 5 & -3 & 1 \\ 3 & 1 & -5 \\ 1 & -2 & 7 \end{vmatrix} = 35 - 6 + 15 - 1 + 63 - 50 = 56 \neq 0.$$

Calculate the auxiliary determinants:

$$\Delta_1 = \begin{vmatrix} 2 & -3 & 1 \\ 4 & 1 & -5 \\ 3 & -2 & 7 \end{vmatrix} = 14 + 45 - 8 - 3 + 84 - 20 = 112,$$

$$\Delta_2 = \begin{vmatrix} 5 & 2 & 1 \\ 3 & 4 & -5 \\ 1 & 3 & 7 \end{vmatrix} = 140 - 10 + 9 - 4 - 42 + 75 = 168,$$

$$\Delta_3 = \begin{vmatrix} 5 & -3 & 2 \\ 3 & 1 & 4 \\ 1 & -2 & 3 \end{vmatrix} = 15 - 12 - 12 - 2 + 27 + 40 = 56.$$

According to the formulas (3.3) we can find the value of system unknowns x_1, x_2, x_3:

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{112}{56} = 2,\ x_2 = \frac{\Delta_2}{\Delta} = \frac{168}{56} = 3,\ x_3 = \frac{\Delta_3}{\Delta} = \frac{56}{56} = 1.$$

Check out:

$$\begin{cases} 5 \cdot 2 - 3 \cdot 3 + 1 = 10 - 9 + 1 = 2; \\ 3 \cdot 2 + 3 - 5 \cdot 1 = 6 + 3 - 5 = 4; \\ 2 - 2 \cdot 3 + 7 \cdot 1 = 2 - 6 + 7 = 3. \end{cases}$$

Answer: $x_1 = 2$, $x_2 = 3$, $x_3 = 1$.

Example 3.3 Solve a square system

$$\begin{cases} 2x_1 + 8x_2 - 7x_3 = 2; \\ 2x_1 - 5x_2 + 6x_3 = 1; \\ 4x_1 + 3x_2 - x_3 = 2. \end{cases}$$

Solution. Calculate the main determinant of the system

$$\begin{vmatrix} 2 & 8 & -7 \\ 2 & -5 & 6 \\ 4 & 3 & -1 \end{vmatrix} = 10 + 192 - 42 - 140 + 16 - 36 = 0.$$

Answer: the given system cannot be solved by the

Cramer’s rule because the system main determinant equals zero.

In this case we have to use other more universal method, which will be examined further.

Questions for self-control

1. What kind of the form has a system of m linear algebraic equations (SLAE) with n unknowns?
2. Which of a system is called the compatible?
3. Which of a system is called the defined system?
4. What is the system of linear algebraic equations which has all free terms are zeros? Does such system have a solution? How many?
5. How can we find a solution of a square SLR with an inverse matrix?
6. How should you do the check out of your solution? What result can you get?
7. How to solve the square system of linear equations by Cramer’s rule?
8. Could the main determinant of the system be zero? What does it mean? What should you do?
9. Could the inverse matrix method and Cramer method be applied to solve any kind of systems? Explain your answer.

Tasks for revision

3.1 Solve the system of equations $\begin{cases} ax_1 + bx_2 = a; \\ bx_1 + ax_2 = b, \end{cases}$ $(a, b \neq 0)$ by two methods: by the inverse matrix method and Cramer’s rule.

3.2 Solve the system of equations by Cramer’s rule

$$\begin{cases} x_2 - 3x_3 + 4x_4 = -5; \\ x_1 - 2x_3 + 3x_4 = -4; \\ 3x_1 + 2x_2 - 5x_4 = 12; \\ 4x_1 + 3x_2 - 5x_3 = 5, \end{cases}$$

4 Matrices elementary transformations. The concept of the rank of the matrix. Kronecker-Capelli theorem. Gaussian elimination method

We select in the matrix A, which has any dimension $m \times n$, k rows and k columns $\left(1 \le k \le \min\{m, n\}\right)$. The determinant composed of elements at the intersection of selected rows, is called *minor* M_k of the k order of the matrix A.

Rank of the matrix A, ($rang\, A$, $r(A)$, $rank\, A$, $rg\, A$), with dimension $m \times n$ is the largest (the highest) order of a non-zero minor of this matrix.

It is clear, that

$$0 \le rang\, A \le \min\{m, n\},$$

at this the rank is zero only for the zero matrix.

The *principal minor* of the matrix A is called an arbitrary non-zero minor, whose order is equal to the rank of the matrix.

A minor M_{k+1} of the $(k+1)$ order, which contains some minor M_k of the k order, is called the *bordering minor* for this minor M_k or the *characteristic minor*.

Definition. A characteristic minor is a minor with $r+1$ rows and $r+1$ columns obtained from the principal minor adding like $r+1$ row the elements from one of the remaining rows and the $r+1$ column is the column of the constant terms.

Theorem 4.1 If a matrix A has non-zero minor $M_r \neq 0$ of the r order, and all its characteristic minor M_{r+1} of the $(r+1)$ order equal zero, then the number r is a rank of the martix A. (Without proof).

Method of the bordering minors to find the rank of the matrix A, which has a dimension as $m \times n$, consists of such steps:

1) choose the minor of the first order M_1, which can be any number at the first row and doesn't equal zero.

2) compute turn by turn the following characteristic minor M_2, which has the order larger than the order of the previous minor. If any minor of this order doesn't equal zero, then it can be taken as a principal minor and we should consider the next minor of a higher order by one. If all of such order characteristic minors equal zero, go to the following step 3.

3) this order of the characteristic minor is a rank of the matrix.

So, this operation should be carried out gradually and considering the minors of all orders.

Example 4.1 Compute the rank of the given matrix A by method of the bordering minors and point its principal minor

$$A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ -2 & -6 & 5 & 5 \\ 2 & 6 & -2 & 4 \end{pmatrix}.$$

Solution. We will consider characteristic minors, starting with the minors of the first order and gradually moving to a higher order minors.

$$M_1 = 1 \neq 0;$$

$$M_2 = \begin{vmatrix} 1 & 3 \\ -2 & -6 \end{vmatrix} = -6 + 6 = 0, \quad M_2 = \begin{vmatrix} 1 & -1 \\ -2 & 5 \end{vmatrix} = 5 - 2 = 3 \neq 0;$$

$$M_3 = \begin{vmatrix} 1 & 3 & -1 \\ -2 & -6 & 5 \\ 2 & 6 & -2 \end{vmatrix} = 12 + 12 + 30 - 12 - 12 - 30 = 0,$$

$$M_3 = \begin{vmatrix} 1 & -1 & 2 \\ -2 & 5 & 5 \\ 2 & -2 & 4 \end{vmatrix} = 20 + 8 - 10 - 20 - 8 + 10 = 0; \text{ so } rang\, A = 2;$$

$$M_2 = \begin{vmatrix} 1 & -1 \\ -2 & 5 \end{vmatrix} \text{ is a principal minor.}$$

Answer: $rang\, A = 2$; $M_2 = \begin{vmatrix} 1 & -1 \\ -2 & 5 \end{vmatrix}$.

Elementary transformations of the matrix are the following operations:

1) swapping of any two rows (columns);

2) multiplying elements of any row by an arbitrary non-zero number;

3) adding to all of elements of any row of the corresponding elements of any other row, multiplied by the same arbitrary number.

Two matrices A i B is called *equivalent* and denote as $A \sim B$, if one of them can be obtained from another due to use elementary transformations.

Theorem 4.2 Equivalent matrices have the same rank

$$A \sim B \Rightarrow rang\, A = rang\, B.$$

In other words, elementary transformations don't change the rank of matrix. (Without proof).

Elementary transformation method of finding the matrix rank consists of the reducing of matrix with dimension $m \times n$ using elementary transformations of the rows and the rearrangement of the columns to the equivalent echelon form matrix $\widetilde{A}$ (upper trapezium or upper triangular):

$$\widetilde{A} = \begin{pmatrix} 1 & \widetilde{a}_{12} & \dots & \widetilde{a}_{1r} & \widetilde{a}_{1(r+1)} & \dots & \widetilde{a}_{1n} \\ 0 & 1 & \dots & \widetilde{a}_{2r} & \widetilde{a}_{2(r+1)} & \dots & \widetilde{a}_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \widetilde{a}_{1(r+1)} & \dots & \widetilde{a}_{rn} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix},$$

in which is non-zero diagonal elements equal unit. Rank of the trapezium form matrix $\widetilde{A}$ equals the number r of its non-zero rows.

Then $rang\, A = rang\, \widetilde{A} = r$.

For the principal minor $\widetilde{M}_r$ of the trapezium form matrix $\widetilde{A}$ we can take the angular minor

$$\widetilde{M}_r = \begin{vmatrix} 1 & \widetilde{a}_{12} & \dots & \widetilde{a}_{1r} \\ 0 & 1 & \dots & \widetilde{a}_{2r} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{vmatrix}.$$

Example 4.2 Compute the rank of the given matrix A by elementary transformation method

$$A = \begin{pmatrix} 2 & 3 & -1 & 2 \\ -3 & -6 & 5 & 5 \\ 5 & 9 & -6 & -3 \end{pmatrix}.$$

Solution. For convenience, we will change the first and third columns:

$$A=\begin{pmatrix}2 & 3 & -1 & 2\\ -3 & -6 & 5 & 5\\ 5 & 9 & -6 & -3\end{pmatrix}\sim\begin{pmatrix}-1 & 3 & 2 & 2\\ 5 & -6 & -3 & 5\\ -6 & 9 & 5 & -3\end{pmatrix}\sim$$

We multiply the first row by 5 and add it to the second row, after it we multiply the first row by (-6) and add it to the third row; as a result we will get the matrix. Note that the first row remains unchanged.

$$\sim\begin{pmatrix}-1 & 3 & 2 & 2\\ 0 & 9 & 7 & 15\\ 0 & -9 & -7 & -15\end{pmatrix}\sim$$

We add the second row to the third row, don't change the second row:

$$\sim\begin{pmatrix}-1 & 3 & 2 & 2\\ 0 & 9 & 7 & 15\\ 0 & 0 & 0 & 0\end{pmatrix}\sim\begin{pmatrix}-1 & 3 & 2 & 2\\ 0 & 1 & 7/9 & 5/3\\ 0 & 0 & 0 & 0\end{pmatrix}\sim\begin{pmatrix}1 & -3 & -2 & -2\\ 0 & 1 & 7/9 & 5/3\\ 0 & 0 & 0 & 0\end{pmatrix};$$

$$rang\,A=r=2\,.$$

Answer: $rang\,A=r=2$.

Kronecker-Capelli theorem. A system of m linear algebraic equations with n unknowns $A\cdot X=B$ is compatible if and only if the rank of the augmented matrix $\tilde{A}$ equals to a rank of the system main matrix A: $rang\,\tilde{A}=rang\,A=r$. In the case of compatibility:

1) if the rank of these matrices is equal to the number of unknowns $r=n$, then the system has a unique solution (is

defined);

2) if this joint rank is less than the number of unknowns $r < n$, then the system is indefinite and has an infinite number of solutions, which depends on $n - r$ arbitrary constants or parameters (Figure 4.3).

(Without proof)

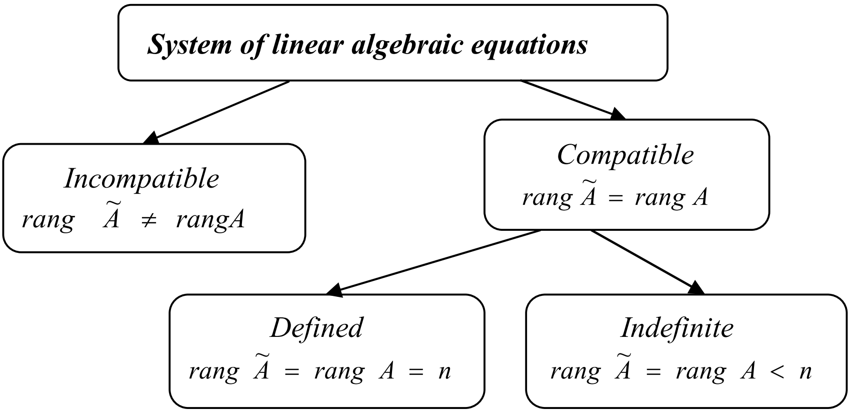


Figure 4.3

Since the augmented matrix $\widetilde{A}$ includes the coefficients matrix A, then $rang\ A \le rang\ \widetilde{A}$. The augmented matrix $\widetilde{A}$ consists of the coefficients martix A and complemented by only one column, so $rang\ \tilde{A} \le rang\ A + 1$.

Let the system be compatible ($rang\ \widetilde{A} = rang\ A = r$) and M_r is some (arbitrarily selected) principal minor of the coefficients matrix A.

If we leave only all those equations in the system, the coefficients of which are included in the principal minor, then the resulting system will be equivalent to the initial.

If compatible system is *indefinite* ($rang\ \tilde{A} = rang\ A = r < n$), then only those r unknowns x_j, whose coefficients are included in the selected principal minor M_r, are called as *principal*, and the rest $n-r$ unknowns x_j are designated as *free* (or independent, or parameters).

We will leave in the system only all those equations, which coefficients of the minor are included in the principal minor, and we will move all the members with free unknowns to the right. Considering free unknowns as arbitrary constants, we will get a square system of the r order regarding principal unknowns, the determinant of which is the principal minor M_r (Appendix A). Since $M_r \neq 0$, then principal minors are found uniquely. Thereby, we get the *general solution* to the initial system. When free unknowns (parameters) are arbitrarily selected fixed values, we obtain a *partial solution*. A partial solution, which is corresponded to zero values of free unknowns, is called the *fundamental solution*.

Example 4.3 Make sure that the system

$$\begin{cases} 3x_1 + x_2 + 2x_3 = 3; \\ 6x_1 + 2x_2 - x_3 = 1; \\ 3x_1 + x_2 - 3x_3 = -2, \end{cases}$$

is compatible and indefinite.

Solution. To establish the system compatibility we need to define the rank of the coefficients matrix and rank of the augmented matrix. To find the rank of the matrix A we use the method of the bordering minors.

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 6 & 2 & -1 \\ 3 & 1 & -3 \end{pmatrix};$$

$$M_1 = 3 \neq 0;$$

$$M_2 = \begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix} = 6 - 6 = 0, \quad M_2 = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 5 = -5 \neq 0;$$

$$M_3 = \begin{vmatrix} 3 & 1 & 2 \\ 6 & 2 & -1 \\ 3 & 1 & -3 \end{vmatrix} = -18 + 12 - 3 - 12 + 18 + 3 = 0;$$

so $rang\, A = 2$; $M_2 = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5 \neq 0$ is the principal minor.

Find out the rank of the augmented matrix $\tilde{A}$ by elementary transformation method. For convenience, we will swap the first column and the second column:

$$\tilde{A} = \left(\begin{array}{ccc|c} 3 & 1 & 2 & 3 \\ 6 & 2 & -1 & 1 \\ 3 & 1 & -3 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 2 & 6 & -1 & 1 \\ 1 & 3 & -3 & -2 \end{array}\right) \sim$$

Multiply the first row by (-2) and add to the second row, then multiply the first row by (-1) and add to the third row, at the result we have the matrix:

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -5 & -5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{array}\right) \sim$$

Add the second row to the third row:

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

we get that $rang\,\tilde{A} = 2$.

Since $rang\ A = rang\ \tilde{A} = r = 2 < n = 3$, then according to Kronecker-Capelli theorem the given system is compatible and indefinite.

Example 4.4 Using Cramer rule, find the general solution of the compatible and undefined system

$$\begin{cases} 3x_1 + x_2 + 2x_3 = 3; \\ 6x_1 + 2x_2 - x_3 = 1; \\ 3x_1 + x_2 - 3x_3 = -2, \end{cases}$$

and extract the fundamental solution from the general solution.

Solution. We accept x_2 and x_3 as principal unknowns (they correspond to the columns of the principal minor, look at the example 4.3.), and x_1 is a free unknown (it corresponds to the column which isn't included in the principal minor). Let $x_1 = C_1$, where C_1 is an arbitrary constant (parameter). We leave in the system only the first and second equations corresponding to the principal minor rows (look at the Appendix A). We move the member with a free unknown $x_1 = C_1$ to the right. We get the square system regarding the principal unknowns x_2 and x_3, and solve it by Cramer's rule:

$$x_1 = C_1\ ; \quad \begin{cases} x_2 + 2x_3 = 3 - 3C_1; \\ 2x_2 - x_3 = 1 - 6C_1, \end{cases}$$

$$\Delta = M_2 = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5;$$

$$\Delta^{(1)} = \begin{vmatrix} 3 - 3C_1 & 2 \\ 1 - 6C_1 & -1 \end{vmatrix} = 15C_1 - 5; \quad \Delta^{(2)} = \begin{vmatrix} 1 & 3 - 3C_1 \\ 2 & 1 - 6C_1 \end{vmatrix} = -5;$$

$$x_2 = \frac{\Delta^{(1)}}{\Delta} = \frac{15C_1 - 5}{-5} = -3C_1 + 1; \quad x_3 = \frac{\Delta^{(2)}}{\Delta} = \frac{-5}{-5} = 1.$$

So, the general solution is:

$$x_1 = C_1,\ x_2 = 1 - 3C_1,\ x_3 = 1,\ C_1 \in R.$$

Let $x_1 = C_1 = 0$. Then the fundamental solution will have a form: $x_1 = 0$, $x_2 = 1$, $x_3 = 1$.

Answer: $x_1 = C_1$, $x_2 = 1 - 3C_1$, $x_3 = 1$, $C_1 \in R$ is the general solution; $x_1 = 0$, $x_2 = 1$, $x_3 = 1$ is the fundamental solution.

Example 4.5 Find out how many solutions does a system of equations that is given by an expanded matrix (or augmented matrix)

$$\tilde{A} = \left(\begin{array}{cccc|c} 1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & a \end{array}\right)$$

at different values of the parameter a have.

Solution. If $a \neq 0$, then $rang\ \tilde{A} = 4$, and $rang\ A = 3$. In this case the system is incompatible and it has no solutions.

If $a = 0$, $rang\ A = rang\ \tilde{A} = 3$, it is less than the number of unknowns, the number of which is equal to four. Then one of the unknowns should be considered as free, and thus the system has a solution for any of its values. Consequently, the system has an infinite number of solutions (compatible and undefined).

Consider the problem of applying the method of elementary transformations of the matrices known as the

Gaussian elimination method to find an arbitrary rectangular SLAE solution.

Suppose we have an arbitrary rectangular system in which m linear algebraic equations and n unknowns x_j $\left(j=\overline{1, n}\right)$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1; \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m. \end{cases}$$

Let us denote A as the coefficients matrix, and $\widetilde{A}$ as an augmented matrix of the given system. A matrix $\widetilde{A}$ composed from the coefficients matrix A due to addition to it of a new column of the system constant terms. Elementary transformations of rows of the augmented matrix $\widetilde{A}$ and swapping of the coefficients matrix A columns correspond to the following equivalent transformations of the linear system which are:

1) swapping any two equations (renumbering of equations);
2) multiplying both parts of any equation by an arbitrary non-zero number;
3) adding to both parts of any equation the corresponding parts of another equation, multiplied by an arbitrary number;
4) renumbering of unknowns.

The research and solution of the SLAE by Gaussian elimination method consists of two stages.

The first stage of Gaussian elimination method (*a straight course*), is when the augmented matrix $\tilde{A}$ due to usage the elementary transformations is turning into a stepwise matrix.

A stepwise matrix (echelon form matrix) is a matrix, elements of which that are under the main diagonal, equal to zero.

In order, to achieve this matrix form, it is necessary to perform sequential extraction of unknowns using the specified equivalent system transformations. First of all, we should pick out the first equation and, accordingly, the first unknown. Assume, that $a_{11} \neq 0$. If this condition is not satisfied, then the equations are rearranged and/or renumbered unknown so that this coefficient is non-zero. Then gradually add the first equation to the second equation, then to the third equation, etc. up to the last, multiplied by some factors. These factors are chosen so that when adding the first coefficients, in each case, zero is obtained. Then the second equation and accordingly the second unknown is allocated. We should repeat this procedure for adding the second equation to all others, while adhering to the advice that when adding other coefficients, in each case, a zero have to be obtained. This process continues until it reaches the last lowest equation or the situation, where the selected equation and all the equations below it, have only zero coefficients for unknowns.

If a matrix has acquired a triangular form during performing those transformations (Figure 4.4), then a system is defined compatible and it has a unique solution. Its solution can be found through carrying out the second stage of Gaussian elimination method (*the inverse course)*. To do this, obtained matrix should be written in the form of a system of linear equations and, after it, we can solve this system moving "from the bottom up", starting with the last equation, which is the common linear equation. Having obtained value of x_k we plug in at the previous upper equation and find a value of x_{k-1} and continue to do the same acts.

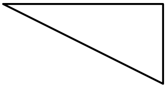


Figure 4.4

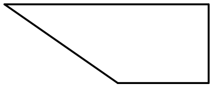


Figure 4.5

If at the result of those transformations matrix has a trapezium form (figure 4.5), then the system is compatible indefinite and it has infinite set of solutions. We reject zero equations (identities $0=0$). All members that contain free unknowns are transferred to the right part. Thus, we get the system of the upper triangular shape (figure 4.4) relative to the principal unknown and solve it, moving from the bottom up. First of all, we should find the principal unknown x_k from the last equation. After this, we put this obtained value x_k at the penultimate equation and define a value x_{k-1} from it, continue to do the same acts until a value of x_1 is found.

Example 4.6 Solve the system of equations from example 3.2 by Gaussian elimination method and compare answers.

Solution. Write down the augmented matrix of the given system:

$$\begin{cases} 5x_1-3x_2+x_3=2; \\ 3x_1+x_2-5x_3=4; \\ x_1-2x_2+7x_3=3, \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 5 & -3 & 1 & 2 \\ 3 & 1 & -5 & 4 \\ 1 & -2 & 7 & 3 \end{array}\right).$$

For convenience we will swap the first and the second rows:

$$\left(\begin{array}{ccc|c} 5 & -3 & 1 & 2 \\ 3 & 1 & -5 & 4 \\ 1 & -2 & 7 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 7 & 3 \\ 3 & 1 & -5 & 4 \\ 5 & -3 & 1 & 2 \end{array}\right) \sim$$

We will exclude the first coefficients from the first column below the first row. For this we will add the first row multiplied by (-3) to the second row and add the first row multiplied by (-5) to the third row. Then we will exclude coefficients from the second column below the second row. At the result we have:

$$\sim\left(\begin{array}{ccc|c} 1 & -2 & 7 & 3 \\ 0 & 1+(-2)\cdot(-3) & -5+7\cdot(-3) & 4+3\cdot(-3) \\ 5 & -3 & 1 & 2 \end{array}\right)\sim$$

$$\sim\left(\begin{array}{ccc|c} 1 & -2 & 7 & 3 \\ 0 & 7 & -26 & -5 \\ 0 & -3+(-2)\cdot(-5) & 1+7\cdot(-5) & 2+3\cdot(-5) \end{array}\right)\sim\left(\begin{array}{ccc|c} 1 & -2 & 7 & 3 \\ 0 & 7 & -26 & -5 \\ 0 & 7 & -34 & -13 \end{array}\right)\sim$$

$$\sim\left(\begin{array}{ccc|c} 1 & -2 & 7 & 3 \\ 0 & 7 & -26 & -5 \\ 0 & -7+7 & 26-34 & -13+5 \end{array}\right)\sim\left(\begin{array}{ccc|c} 1 & -2 & 7 & 3 \\ 0 & 7 & -26 & -5 \\ 0 & 0 & -8 & -8 \end{array}\right).$$

The obtained matrix has a triangular form, thus, our system will have a unique solution, which can be found and let's do it. Let us make and solve a system of equations:

$$\begin{cases} x_1-2x_2+7x_3=3; \\ 7x_2-26x_3=-5; \\ 8x_3=8, \end{cases} \Rightarrow \begin{cases} x_1-2x_2+7x_3=3; \\ 7x_2-26x_3=-5; \\ x_3=1, \end{cases} \Rightarrow \begin{cases} x_1-2x_2+7x_3=3; \\ 7x_2-26\cdot 1=-5; \\ x_3=1, \end{cases} \Rightarrow$$

$$\begin{cases} x_1-2x_2+7x_3=3; \\ 7x_2=21; \\ x_3=1, \end{cases} \Rightarrow \begin{cases} x_1-2\cdot 3+7=3; \\ x_2=3; \\ x_3=1, \end{cases} \Rightarrow \begin{cases} x_1=2; \\ x_2=3; \\ x_3=1. \end{cases}$$

As we can see, our answers coincided with previously obtained values. (Conclusion to be made by yourself).

Answer: $x_1 = 2$, $x_2 = 3$, $x_3 = 1$.

Example 4.7 Solve the system of linear algebraic equations:

$$\begin{cases} x_1 - 4x_2 + 4x_3 - x_5 = 0; \\ -x_1 + 2x_2 - x_3 - x_4 + 2x_5 = 1; \\ -2x_2 + x_3 + 2x_5 = 1; \\ -2x_1 + 2x_2 - 4x_3 - x_4 + 4x_5 = -1. \end{cases}$$

Solution. Write our system as the augmented matrix:

$$\tilde{A} = \left(\begin{array}{ccccc|c} 1 & -4 & 4 & 0 & -1 & 0 \\ -1 & 2 & -1 & -1 & 2 & 1 \\ 0 & -2 & 1 & 0 & 2 & 1 \\ -2 & 2 & -4 & -1 & 4 & -1 \end{array}\right).$$

Get zeros at the first column. For this we add the elements from the first row to the corresponding elements from the second row, after it we multiply the first row by 2 and add to the fourth row:

$$\tilde{A} = \left(\begin{array}{ccccc|c} 1 & -4 & 4 & 0 & -1 & 0 \\ 0 & -2 & 3 & -1 & 1 & 1 \\ 0 & -2 & 1 & 0 & 2 & 1 \\ 0 & -6 & 4 & -1 & 2 & -1 \end{array}\right).$$

Now, we will get zeros at the second column, for this we will do the following: multiply the second row by (-1) and add to the third, and then multiply the second row by (-3) and add to the fourth row:

$$\left(\begin{array}{ccccc|c} 1 & -4 & 4 & 0 & -1 & 0 \\ 0 & -2 & 3 & -1 & 1 & 1 \\ 0 & 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & -5 & 2 & -1 & -4 \end{array}\right)$$

Next, we will swap the third and fifth columns:

$$\begin{array}{c} \begin{array}{cccccc} & & x_5 & & x_3 & \end{array} \\ \left(\begin{array}{ccccc|c} 1 & -4 & -1 & 0 & 4 & 0 \\ 0 & -2 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & -1 & 2 & -5 & -4 \end{array}\right) \end{array}$$

Further, we will get zeros at the third row, for this: add the elements from the third row to the corresponding elements from the fourth row:

$$\begin{array}{c} \begin{array}{cccccc} & & x_5 & & x_3 & \end{array} \\ \left(\begin{array}{ccccc|c} 1 & -4 & -1 & 0 & 4 & 0 \\ 0 & -2 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 3 & -7 & -4 \end{array}\right) \end{array}.$$

At the result of those transformations, our matrix has acquired a trapezium form (Figure 4.5), that means that the given system is indefinite, compatible and it has unlimited number of solutions.

System can be rewritten as: $\begin{cases} x_1 - 4x_2 - x_5 + 4x_3 = 0; \\ -2x_2 + x_5 - x_4 + 3x_3 = 1; \\ x_5 + x_4 - 2x_3 = 0; \\ 3x_4 - 7x_3 = -4. \end{cases}$

Let $x_3 = C_1$, where C_1 is arbitrary constant. Move all members with a free unknown $x_3 = C_1$ to the right. We get the system of the upper triangular form (Figure 4.4) regarding the principal unknowns (look an example at the appendix A) x_1, x_2, x_4, x_5 and solve this system, moving from the bottom up:

$$\begin{cases} x_1 - 4x_2 - x_5 = -4C_1; \\ -2x_2 + x_5 - x_4 = 1 - 3C_1; \\ x_5 + x_4 = 2C_1; \\ 3x_4 = -4 + 7C_1, \end{cases}$$

$$\begin{cases} x_1 = -4C_1 + 4x_2 + x_5; \\ -2x_2 = 1 - 3C_1 - x_5 + x_4; \\ x_5 = 2C_1 - x_4; \\ x_4 = \dfrac{1}{3}(7C_1 - 4), \end{cases} \qquad \begin{cases} x_1 = -4C_1 + 4x_2 + x_5; \\ 2x_2 = 3C_1 + x_5 - x_4 - 1; \\ x_5 = 2C_1 - \dfrac{7}{3}C_1 + \dfrac{4}{3} = -\dfrac{1}{3}C_1 + \dfrac{4}{3}; \\ x_4 = \dfrac{7}{3}C_1 - \dfrac{4}{3}, \end{cases}$$

$$\begin{cases} x_1 = -4C_1 + 4x_2 + x_5; \\ x_2 = \dfrac{1}{2}\left(3C_1 - \dfrac{1}{3}C_1 + \dfrac{4}{3} - \dfrac{7}{3}C_1 + \dfrac{4}{3} - 1\right) = \dfrac{1}{2}\left(\dfrac{1}{3}C_1 + \dfrac{5}{3}\right) = \dfrac{1}{6}C_1 + \dfrac{5}{6}; \\ x_5 = -\dfrac{1}{3}C_1 + \dfrac{4}{3}; \\ x_4 = \dfrac{7}{3}C_1 - \dfrac{4}{3}, \end{cases}$$

$$\begin{cases} x_1 = -4C_1 + \frac{2}{3}C_1 + \frac{10}{3} - \frac{1}{3}C_1 + \frac{4}{3} = -\frac{11}{3}C_1 + \frac{14}{3}; \\ x_2 = \frac{1}{6}C_1 + \frac{5}{6}; \\ x_5 = -\frac{1}{3}C_1 + \frac{4}{3}; \\ x_4 = \frac{7}{3}C_1 - \frac{4}{3}, \end{cases}$$

$$\begin{cases} x_1 = -\frac{11}{3}C_1 + \frac{14}{3}; \\ x_2 = \frac{1}{6}C_1 + \frac{5}{6}; \\ x_3 = C_1; \\ x_4 = \frac{7}{3}C_1 - \frac{4}{3}; \\ x_5 = -\frac{1}{3}C_1 + \frac{4}{3}. \end{cases}$$

Let $x_3 = 6t$, $t \in R$, then the general solution will be as the following:

$$x_1 = -22t + \frac{14}{3},\ x_2 = t + \frac{5}{6},\ x_3 = 6t,\ x_4 = 14t - \frac{4}{3},\ x_5 = -2t + \frac{4}{3}.$$

We will check the obtained solution:

$$\begin{cases} -22t+\dfrac{14}{3}-4\cdot\left(t+\dfrac{5}{6}\right)-\left(-2t+\dfrac{4}{3}\right)+4\cdot 6t=0; \\ -2\cdot\left(t+\dfrac{5}{6}\right)-2t+\dfrac{4}{3}-\left(14t-\dfrac{4}{3}\right)+3\cdot 6t=1; \\ -2t+\dfrac{4}{3}+14t-\dfrac{4}{3}-2\cdot 6t=0; \\ 3\cdot\left(14t-\dfrac{4}{3}\right)-7\cdot 6t=-4, \end{cases}$$

$$\begin{cases} -22t+\dfrac{14}{3}-4t-\dfrac{10}{3}+2t-\dfrac{4}{3}+24t=\dfrac{14-10-4}{3}=0; \\ -2t-\dfrac{5}{3}-2t+\dfrac{4}{3}-14t+\dfrac{4}{3}+18t=\dfrac{-5+4+4}{3}=1; \\ -2t+\dfrac{4}{3}+14t-\dfrac{4}{3}-12t=\dfrac{4}{3}-\dfrac{4}{3}=0; \\ 42t-4-42t=-4. \end{cases}$$

Answer: $x_1=\dfrac{14}{3}-22t$, $x_2=\dfrac{5}{6}+t$, $x_3=6t$, $x_4=14t-\dfrac{4}{3}$, $x_5=\dfrac{4}{3}-2t$, $t\in R$.

Questions for self-control

1. What is the matrix rank?
2. How can we calculate the rank of the matrix by the method of bordering minors?
3. What operations are called elementary transformations of the matrix?
4. What matrices are called equivalent?
5. How can we calculate the rank of the matrix by method of elementary transformations?

6. Formulate the Kronecker-Capelli theorem for linear systems.

7. Does the system have a solution if the rank of the augmented matrix is greater than the rank of the basic matrix?

8. How is an arbitrary SLAE solved by the Gaussian elimination method?

9. Is it possible to determine the consistency of the system using the Gaussian elimination method? Explain your answer.

10. Tell the sequences of operations of the straight course of Gaussian elimination.

11. How can we know by performing the Gaussian elimination method that the system does not have a solution?

12. What is the difference between the fundamental solution and the general solution? How should we find them?

Tasks for revision

4.1 Find out at which parameters a and b the system will be the compatible.

$$\begin{cases} x_1 + 2x_2 + 3x_3 = a; \\ 4x_1 + 5x_2 + 6x_3 = b; \\ 7x_1 + 8x_2 + 9x_3 = 0. \end{cases}$$

4.2 Solve the system of linear algebraic equations using the one of the learned methods

$$\begin{cases} x_1 - x_2 + 2x_3 + x_4 = 3; \\ 2x_2 + 3x_3 - x_4 = 4; \\ 2x_1 - x_3 + 5x_4 = 6; \\ x_1 + x_2 - x_3 - x_4 = 0. \end{cases}$$

4.3 Solve the system of linear algebraic equations using the Cramer’s rule and Gauss elimination method, and compare obtained results

$$\begin{cases} 2x_1 - 3x_2 - x_3 = -9; \\ x_1 + 2x_2 + x_3 = 4; \\ -3x_1 + 2x_2 - 4x_3 = 3. \end{cases}$$

4.4 Solve the system of equations by Gaussian elimination method

$$\begin{cases} x + 2y = 5; \\ 3y + 4z = 18; \\ 5z + 6u = 39; \\ 7u + 8v = 68; \\ 9v + 10x = 55. \end{cases}$$

4.5 Explore the system of equations for compatibility

$$\begin{cases} x_1 + 3x_2 + 5x_3 + 7x_4 + 9x_5 = 1; \\ x_1 - 2x_2 + 3x_3 - 4x_4 + 5x_5 =; \\ 2x_1 + 11x_2 + 12x_3 + 25x_4 + 22x_5 = 4. \end{cases}$$

5 Gauss- Jordan elimination method

Gauss-Jordan elimination method is based on elementary transformations of the augmented matrix rows

$$\tilde{A} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right)$$

of the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1; \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m. \end{cases}$$

At the result of each of elementary transformations the augmented matrix is changing, however, the system of equation corresponded to the obtained matrices is equivalent to the initial system.

Suppose the system of m linear equations with n unknowns is given. Using the elementary transformations, we will build an equivalent system of a special form. As the first equation, we choose the one in which the coefficient of x_1 is not equal to zero. Suppose that $a_{11} \neq 0$, then the first system is

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1.$$

Multiply the first row by $\dfrac{1}{a_{11}}$. Then multiply the first row by $(-a_{i1})$, $(i = 2, 3, \ldots, m)$ and add it term by term to the equations with numbers as $i = 2, 3, \ldots, m$. After these transformations at the equation with numbers $i \geq 2$ we will provide the exclusion of x_1. The first step of Gauss-Jordan elimination method is finished. The equivalent augmented matrix has a form:

$$\tilde{A}^{(1)} = \left(\begin{array}{cccc|c} 1 & a_{12}^{(1)} & \ldots & a_{1n}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(1)} & \ldots & a_{2n}^{(1)} & b_2^{(1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & a_{m2}^{(1)} & \ldots & a_{mn}^{(1)} & b_m^{(1)} \end{array}\right).$$

It may happen that at the first step, the variable x_1 and unknowns $x_2, x_3, ..., x_{j_{k-1}}$ ($j_{k-1} < n$), will be excluded together, but there will be at least one of the equation in which the unknown x_{j_k} remains. One of these equations will be taken as the second equation of the system. In this case, an augmented matrix $\tilde{A}^{(1)}$ corresponding to the resulting system will look like

$$\tilde{A}^{(1)} = \left(\begin{array}{cccccc|c} 1 & a_{12}^{(1)} & ... & a_{1j_k}^{(1)} & ... & a_{1n}^{(1)} & b_1^{(1)} \\ 0 & 0 & ... & a_{2j_k}^{(1)} & ... & a_{2n}^{(1)} & b_2^{(1)} \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & a_{mj_k}^{(1)} & ... & a_{mn}^{(1)} & b_m^{(1)} \end{array}\right).$$

We use the second equation to exclude an unknown x_{j_k} of all equations except the second one. After the second step of Gauss- Jordan elimination method we get the equivalent matrix $\tilde{A}^{(2)}$ which looks like

$$\tilde{A}^{(2)} = \left(\begin{array}{ccccccc|c} 1 & a_{12}^{(2)} & ... & 0 & a_{1j_{k+1}}^{(2)} & ... & a_{1n}^{(2)} & b_1^{(2)} \\ 0 & 0 & ... & 1 & a_{2j_{k+1}}^{(2)} & ... & a_{2n}^{(2)} & b_2^{(2)} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & a_{mj_{k+1}}^{(2)} & ... & a_{mn}^{(2)} & b_m^{(2)} \end{array}\right).$$

Continuing the process, after r steps we get a matrix $\tilde{A}^{(r)}$ containing r unit columns in a place of the first n columns of the matrix A (r is a rank of the matrix A of the system).

There are three possible cases:

1. If $rang(A) = rang(\tilde{A}) = n$, the matrix $\tilde{A}$ transformed into a matrix

$$\tilde{A}^{(n)}=\left(\begin{array}{ccccc|c} 1 & 0 & 0 & \ldots & 0 & b_1^{(\mathrm{n})} \\ 0 & 1 & 0 & \ldots & 0 & b_2^{(\mathrm{n})} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & 0 & \ldots & 1 & b_n^{(\mathrm{n})} \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{array}\right).$$

The system has a unique solution:

$$x_1=b_1^{(n)},\ x_2=b_2^{(n)},\ldots,\ x_n=b_n^{(n)}.$$

2. If $rang(A)=rang(\tilde{A})=r$ and $r<n$, then

$$\tilde{A}^{(r)}=\left(\begin{array}{cccccccc|c} 1 & 0 & 0 & \ldots & 0 & a_{1,r+1}^{(r)} & \ldots & a_{1n}^{(r)} & b_1^{(r)} \\ 0 & 1 & 0 & \ldots & 0 & a_{2,r+1}^{(r)} & \ldots & a_{1n}^{(r)} & b_2^{(r)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & a_{r,r+1}^{(r)} & \ldots & a_{rn}^{(r)} & b_r^{(r)} \\ 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \end{array}\right).$$

The system has an infinite set of solutions. The general solution has a form as

$$x_1=b_1^{(r)}-a_{1,r+1}^{(r)}x_{r+1}-\ldots-a_{1n}^{(r)}x_n;$$

$$x_2=b_2^{(r)}-a_{2,r+1}^{(r)}x_{r+1}-\ldots-a_{2n}^{(r)}x_n;$$

....

$$x_r=b_r^{(r)}-a_{r,r+1}^{(r)}x_{r+1}-\ldots-a_{rn}^{(r)}x_n.$$

Unknowns $x_1, x_2, ..., x_r$ are called *principal unknowns,* and unknowns $x_{r+1}, x_{r+2}, ..., x_n$ are called *free unknowns*.

Free variables $x_{r+1}, x_{r+2}, ..., x_n$ can acquire any values, while the principal variables $x_1, x_2, ..., x_r$ will depend on these values. At the result we get unlimited set of the particular solutions.

Among the partial solutions of the system, we select the basic solutions that are obtained if all free unknown systems are zero

$$x_1 = b_1^{(r)},\ x_2 = b_2^{(r)}, ...,\ x_r = b_r^{(r)},\ x_{r+1} = 0, ...,\ x_n = 0 .$$

In the general case, the number of basic solutions does not exceed C_n^r.

3. If $rang(A) \neq rang(\tilde{A})$, then

$$\tilde{A}^{(r)} = \left(\begin{array}{cccccccc|c} 1 & 0 & 0 & ... & 0 & a_{1,r+1}^{(r)} & ... & a_{1n}^{(r)} & b_1^{(r)} \\ 0 & 1 & 0 & ... & 0 & a_{2,r+1}^{(r)} & ... & a_{1n}^{(r)} & b_2^{(r)} \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 & a_{r,r+1}^{(r)} & ... & a_{rn}^{(r)} & b_r^{(r)} \\ 0 & 0 & 0 & ... & 0 & 0 & ... & 0 & b_{r+1}^{(r)} \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 & 0 & ... & 0 & b_n^{(r)} \end{array}\right),$$

where at least one of the elements b_i^r, $r+1 \leq i \leq m$, is different from zero. In this case the system is incompatible.

Thus, Gauss-Jordan elimination method consists of r iterations (r steps). On each S iteration we select the guiding element $a_{i_S, j_S}^{(S-1)} \neq 0$, where i_S, j_S are respectively, the guide row and the column. Due to elementary transformations, the column j_S is converted into a unit column with a unit at the row i_S.

Example 5.1 Solve the system of linear equations

$$\begin{cases} 2x_1+5x_2+4x_3+x_4=20; \\ 2x_1+10x_2+9x_3+7x_4=40; \\ x_1+3x_2+2x_3+x_4=11; \\ 3x_1+8x_2+9x_3+2x_4=37 \end{cases}$$

by Gauss-Jordan elimination method.

Solution. Write down the augmented matrix of the given system of equations:

$$\tilde{A}=\left(\begin{array}{cccc|c} 2 & 5 & 4 & 1 & 20 \\ 2 & 10 & 9 & 7 & 40 \\ 1 & 3 & 2 & 1 & 11 \\ 3 & 8 & 9 & 2 & 37 \end{array}\right).$$

The first step. Swap the first row and the third row:

$$\tilde{A}=\left(\begin{array}{cccc|c} 1 & 3 & 2 & 1 & 11 \\ 2 & 10 & 9 & 7 & 40 \\ 2 & 5 & 4 & 1 & 20 \\ 3 & 8 & 9 & 2 & 37 \end{array}\right).$$

Multiply the first row by (-2) and add to the second row and to the third row, then multiply it by (-3) and add to the fourth row. We get the equivalent the augmented matrix of the system:

$$\tilde{A}^{(1)} = \left(\begin{array}{cccc|c} 1 & 3 & 2 & 1 & 11 \\ 0 & 4 & 5 & 5 & 18 \\ 0 & -1 & 0 & -1 & -2 \\ 0 & -1 & 3 & -1 & 4 \end{array}\right).$$

The second step. Swap the second row and the third row:

$$\tilde{A}^{(1)} = \left(\begin{array}{cccc|c} 1 & 3 & 2 & 1 & 11 \\ 0 & -1 & 0 & -1 & -2 \\ 0 & 4 & 5 & 5 & 18 \\ 0 & -1 & 3 & -1 & 4 \end{array}\right).$$

Multiply the second row by 4 and add to the third row, then multiply the second row by (-1) and add to the fourth:

$$\tilde{A}^{(2)} = \left(\begin{array}{cccc|c} 1 & 3 & 2 & 1 & 11 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 5 & 1 & 10 \\ 0 & 0 & 3 & 0 & 6 \end{array}\right).$$

The third step. Swap the third row and the fourth row:

$$\tilde{A}^{(2)} = \left(\begin{array}{cccc|c} 1 & 3 & 2 & 1 & 11 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 3 & 0 & 6 \\ 0 & 0 & 5 & 1 & 10 \end{array}\right).$$

Multiply the third row by $\frac{1}{3}$, then multiply the third row by (-5) and add to the fourth, we get:

$$\tilde{A}^{(3)} = \left(\begin{array}{cccc|c} 1 & 3 & 2 & 1 & 11 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

The forth step. Multiply the fourth row by (-1) and add to the second row, after it add the result to the first row:

$$\tilde{A}^{(4)} = \left(\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 11 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

The fifth step. Multiply the third row by (-2) and add to the first row:

$$\tilde{A}^{(5)} = \left(\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

The sixth step. Multiply the second row by (-3) and add to the first row:

$$\tilde{A}^{(6)} = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

So, the solution of the system is: $x_1 = 1$, $x_2 = 2$, $x_3 = 2$, $x_4 = 0$.

Check out: $\begin{cases} 2\cdot 1+5\cdot 2+4\cdot 2+0=2+10+8+0=20; \\ 2\cdot 1+10\cdot 2+9\cdot 2+7\cdot 0=2+20+18+0=40; \\ 1+3\cdot 2+2\cdot 2+0=1+6+4+0=11; \\ 3\cdot 1+8\cdot 2+9\cdot 2+2\cdot 0=3+16+18+0=37. \end{cases}$

Answer: $x_1=1$, $x_2=2$, $x_3=2$, $x_4=0$.

Example 5.2 Solve the system of linear equations

$$\begin{cases} x_1+2x_2+3x_3-x_4=0; \\ x_1-x_2+x_3+2x_4=4; \\ x_1+5x_2+5x_3-4x_4=-4; \\ x_1+8x_2+7x_3-7x_4=-8 \end{cases}$$

by Gauss-Jordan elimination method.

Solution. Write down the augmented matrix of the system:

$$\tilde{A}=\left(\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 0 \\ 1 & -1 & 1 & 2 & 4 \\ 1 & 5 & 5 & -4 & -4 \\ 1 & 8 & 7 & -7 & -8 \end{array}\right).$$

The first step. Multiply the first row by (-1) and add to the second row. We get the equivalent augmented matrix:

$$\tilde{A}^{(1)}=\left(\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 0 \\ 0 & -3 & -2 & 3 & 4 \\ 0 & 3 & 2 & -3 & -4 \\ 0 & 6 & 4 & -6 & -8 \end{array}\right).$$

The second step. Add the second row to the third row, then multiply the second row by 2 and add to the fourth row:

$$\tilde{A}^{(2)} = \left(\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 0 \\ 0 & -3 & -2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The third step. Divide the second row by (-3), then multiply the second row by (-2) and add to the first row, and we have:

$$\tilde{A}^{(3)} = \left(\begin{array}{cccc|c} 1 & 0 & 5/3 & 1 & 8/3 \\ 0 & 1 & 2/3 & -1 & -4/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The matrix $\tilde{A}^{(3)}$ defines the general solution of the system:

$$x_1 = \frac{8}{3} - \frac{5}{3}x_3 - x_4, \; x_2 = -\frac{4}{3} - \frac{2}{3}x_3 + x_4,$$

where x_1, x_2 are principal unknowns, x_3, x_4 are free unknowns. We get one of the particular solution, let $x_3 = 0$, $x_4 = 0$, then $x_1 = \frac{8}{3}$, $x_2 = -\frac{4}{3}$.

Check out: $$\begin{cases} \frac{8}{3}+2\cdot\frac{-4}{3}+0-0=\frac{8}{3}-\frac{8}{3}=0; \\ \frac{8}{3}+\frac{4}{3}+0+0=\frac{12}{3}=4; \\ \frac{8}{3}+5\cdot\frac{-4}{3}+0-0=\frac{8}{3}-\frac{20}{3}=-\frac{12}{3}=-4; \\ \frac{8}{3}+8\cdot\frac{-4}{3}+0-0=\frac{8}{3}-\frac{32}{3}=-\frac{24}{3}=-8. \end{cases}$$

Answer: $x_1=\frac{8}{3}-\frac{5}{3}x_3-x_4$, $x_2=-\frac{4}{3}-\frac{2}{3}x_3+x_4$.

Example 5.3 Solve the system of linear equations

$$\begin{cases} 2x_1-x_2-3x_3+4x_4=5; \\ 2x_1-x_2+x_3-x_4=3; \\ 4x_1-2x_2+10x_3-12x_4=2; \\ 4x_1-2x_2-2x_3+3x_4=2 \end{cases}$$

by Gauss-Jordan elimination method.

Solution. Write down the system augmented matrix

$$\tilde{A}=\left(\begin{array}{cccc|c} 2 & -1 & -3 & 4 & 5 \\ 2 & -1 & 1 & -1 & 3 \\ 4 & -2 & 10 & -12 & 2 \\ 4 & -2 & -2 & 3 & 2 \end{array}\right).$$

The first step. Multiply the first row by (-1) and add to the second, then multiply the first row by (-2) and add to the third row and fourth row, we get:

$$\tilde{A}^{(1)} = \left(\begin{array}{cccc|c} 2 & -1 & -3 & 4 & 5 \\ 0 & 0 & 4 & -5 & -2 \\ 0 & 0 & 16 & -20 & -8 \\ 0 & 0 & 4 & -5 & -8 \end{array}\right).$$

The second step. Multiply the second row by (-4) and add to the third row, then multiply the second row by (-1) and add to the fourth row, we get:

$$\tilde{A}^{(1)} = \left(\begin{array}{cccc|c} 2 & -1 & -3 & 4 & 5 \\ 0 & 0 & 4 & -5 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right).$$

The system is incompatible, because the obtained equation $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = -6$ from the fourth row doesn't have a solution at whatever values of the variables x_1, x_2, x_3, x_4.

Answer: the given system is incompatible.

Questions for self-control

1. What is the basis of Gauss-Jordan elimination method?
2. Is it possible to use Gauss-Jordan method to establish the consistency of the linear system?
3. What is common and different in Gaussian elimination method and Gauss-Jordan elimination method?
4. What is the difference between the basic variables and free variables?
5. In which case the system of equations has an infinite set of solutions?

6. How to distinguish basic solutions among partial solutions?

Tasks for revision

5.1 Compute the rank of a matrix by Gauss-Jordan elimination method

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 1 & 3 \\ 3 & 1 & 2 & 5 \\ 2 & 2 & 2 & 3 \end{pmatrix}.$$

5.2 Solve the given system by Gauss-Jordan elimination method

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 14; \\ 3x_1 + 2x_2 + x_3 = 10; \\ x_1 + x_2 + x_3 = 6; \\ 2x_1 + 3x_2 - x_3 = 5; \\ x_1 + x_2 = 3. \end{cases}$$

5.3 Solve the given system by Gauss-Jordan elimination method

$$\begin{cases} x_1 - x_2 + x_3 - x_4 = -2; \\ x_1 + 2x_2 - 2x_3 - x_4 = -5; \\ 2x_1 - x_2 - 3x_3 + 2x_4 = -1; \\ x_1 + 2x_2 + 3x_3 - 6x_4 = -10. \end{cases}$$

6 Application of Gauss-Jordan elimination method for finding the inverse matrix

Consider the application of Gauss-Jordan elimination method to find the inverse matrix A^{-1}. Take the equation $A \cdot X = E$, where A, X are square matrices with a dimension $n \times n$, and E is the unit matrix. Obviously, that matrix equation $A \cdot X = E$ has a unique solution $X = A^{-1}$. The solution of given matrix equation is reduced to a solution of n systems with n linear equations to n unknowns looks like

$$a_{i1}x_{1j} + a_{i2}x_{2j} + \ldots + a_{in}x_{nj} = e_{ij}, \; i, j = \overline{1, n},$$

where $e_{ij} = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$ This system of equations corresponds to the augmented matrix $\tilde{A} = (A|E)$. Applying Gauss-Jordan elimination method to the matrix $\tilde{A}$, we get matrix $\tilde{A}^{(n)} = (E|B)$. Let us show that $B = A^{-1}$. The augmented matrix $\tilde{A}^{(n)}$ corresponds to the matrix equation $E \cdot X = B$, which has a unique solution $X = B$. We obtained this matrix $\tilde{A}^{(n)} = (E|B)$ from the matrix $\tilde{A} = (A|E)$ by Gauss-Jordan elimination method. Therefore, systems of linear equations that correspond to matrices $(E|B)$ i $(A|E)$, are equivalent, i.e., they have the same solution. It follows that $B = A^{-1}$, that is $\tilde{A}^{(n)} = (E|A^{-1})$.

So, to calculate an inverse matrix A^{-1}, for a non-singular matrix A, it is necessary to write down the augmented matrix $\tilde{A} = (A|E)$, then at the matrix $\tilde{A}$ we should transform the matrix A into a unite matrix by Gauss-Jordan elimination method, and then we get an inverse A^{-1} on the place of the unit matrix E.

Example 6.1 Calculate the inverse matrix by Gauss-Jordan elimination method for this matrix

$$A=\begin{pmatrix} 1 & 2 & -1 \\ -1 & 6 & -1 \\ 2 & 6 & 12 \end{pmatrix}.$$

Solution. Write down the augmented matrix

$$\tilde{A}=\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -1 & 6 & -1 & 0 & 1 & 0 \\ 2 & 6 & 12 & 0 & 0 & 1 \end{array}\right).$$

The first step. Add the first row to the second row, then the first row multiplied by (-2) and add it to the third row, we get:

$$\tilde{A}^{(1)}=\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 8 & -2 & 1 & 1 & 0 \\ 0 & 2 & 14 & -2 & 0 & 1 \end{array}\right)$$

The second step. Multiply the second row by $\frac{1}{8}$, then multiply the second row by(-2) and add to the third row:

$$\tilde{A}^{(2)}=\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1/4 & 1/8 & 1/8 & 0 \\ 0 & 2 & 14 & -2 & 0 & 1 \end{array}\right)=\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1/4 & 1/8 & 1/8 & 0 \\ 0 & 0 & 29/2 & -9/4 & -1/4 & 1 \end{array}\right).$$

The third step. Multiply the third row multiplied by $\frac{2}{29}$; multiply the third row by $\frac{1}{4}$ and add to the second row; the third row add to the first row:

$$\tilde{A}^{(3)} = \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1/4 & 1/8 & 1/8 & 0 \\ 0 & 0 & 1 & -9/58 & -1/58 & 2/29 \end{array}\right) =$$

$$= \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 49/58 & -1/58 & 2/29 \\ 0 & 1 & 0 & 5/58 & 7/58 & 1/58 \\ 0 & 0 & 1 & -9/58 & -1/58 & 2/29 \end{array}\right).$$

The fourth step. Multiply the second row by (-2) and add to the first row:

$$\tilde{A}^{(4)} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 39/58 & -15/58 & 1/29 \\ 0 & 1 & 0 & 5/58 & 7/58 & 1/58 \\ 0 & 0 & 1 & -9/58 & -1/58 & 2/29 \end{array}\right).$$

Thus, the inverse matrix A^{-1} has the form:

$$A^{-1} = \begin{pmatrix} 39/58 & -15/58 & 1/29 \\ 5/58 & 7/58 & 1/58 \\ -9/58 & -1/58 & 2/29 \end{pmatrix} \text{ or } A^{-1} = \frac{1}{116}\begin{pmatrix} 78 & -30 & 4 \\ 10 & 14 & 2 \\ -18 & -2 & 8 \end{pmatrix}.$$

Check out:

$$A^{-1} \cdot A = \frac{1}{116}\begin{pmatrix} 78 & -30 & 4 \\ 10 & 14 & 2 \\ -18 & -2 & 8 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & -1 \\ -1 & 6 & -1 \\ 2 & 6 & 12 \end{pmatrix} =$$

$$= \frac{1}{116}\begin{pmatrix} 78+30+8 & 156-180+24 & -78+30+48 \\ 10-14+4 & 20+84+12 & -10-14+24 \\ -18+2+16 & -36-12+48 & 18+2+96 \end{pmatrix} =$$

$$=\frac{1}{116}\begin{pmatrix}116 & 0 & 0\\ 0 & 116 & 0\\ 0 & 0 & 116\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=E\,.$$

So, the inverse matrix was found correctly.

Answer: $A^{-1}=\frac{1}{116}\begin{pmatrix}78 & -30 & 4\\ 10 & 14 & 2\\ -18 & -2 & 8\end{pmatrix}$.

Questions for self-control

1. In which case can you use Gauss-Jordan method except for the solution of systems?
2. How to find the inverse matrix using Gauss-Jordan method?
3. What are the differences between the algorithm for finding the inverse matrix and applying the method for this purpose?
4. What transformations do we use in Gauss-Jordan method for the search of the inverse matrix? Give examples.
5. Could we find the inverse matrix for any matrix? Why?

Tasks for revision

6.1 Find the matrix inverse to the given matrix using Gauss-Jordan method

$$A=\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\\ 3 & 1 & 2\end{pmatrix},.$$

6.2 Find the matrix inverse to the given matrix using Gauss-Jordan method

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

6.3 Find the matrix inverse to the matrix using Gauss-Jordan elimination method

$$A = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

6.4 Find the matrix inverse to the matrix using Gauss-Jordan elimination method

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

7 Linear models of "input-output" and their application for the analysis of the enterprise economic activity

*The model «*input-output*»,* which is also called as a model inter-industry balance by Leontief, it is a background of many linear models of the manufacturing sector of the economy. Really, an input-output model is a quantitative economic technique that represents the interdependencies between different branches of a national economy or different regional economies. Vassily Leontief (1906–1999) is credited with developing of this analysis type and earned the Nobel

Prize in Economics for this model in 1973.

The purpose of balance analysis is the search of the answer to the question of the macroeconomic level associated with the efficiency of conducting a multi-sectoral economy. Namely, what should be the volume of production of each of the n industries to satisfy the needs of products in it. The connection between different industries, which may be both producers of some products and consumers of a particular product or service, is often provided in the form of inter-industry balance sheets. In order to analyze them in 1938, economist V. Leontief work out his mathematical model.

Consider the process of production (provision of services) for a certain period of time and introduce some designations: x_i is the total (gross) volume of production i industry ($i = 1,2,...,n$); x_{ij} is the volume of production of this industry consumed in the production process j industry ($i, j = 1,2,...,n$); y_i is the volume of the final product of the i industry, which is not used for production. Since the gross volume of production of any one i industry is equal to the total volume of products consumed by industries and the final product, then

$$x_i = \sum_{j=1}^{n} x_{ij} + y_i \text{ , } (i = 1,2,...,n),$$

which is called the balance sheet ratio.

The balance sheet ratio can be presented as:

a) in a form $x_i = \sum_{j=1}^{n} a_{ij} x_{ij} + y_i \quad (i = 1,2,...,n)$ (3)

where $a_{ij} = \dfrac{x_{ij}}{x_j}$ ($i, j = 1,2,...,n$) are *coefficients of direct expenses*, which indicate the expenses of products of i industry for production of per unit of j industry;

Note 7.1 If the coefficient a_{ij} is a constant and isn't dependent on the established production technology, then it will point out the linear dependence of material expenses on gross output, that is

$$x_{ij} = a_{ij} x_j, \ (i, j = 1,2,...,n).$$

As a result, the inter-industry balance will be called *linear*.

b) in a matrix form

$$X = AX + Y$$

or $$(E - A)X = Y, \tag{4}$$

where $X = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{pmatrix}$,

X is a vector of gross output, Y is a vector of the final product, A is a matrix of direct expenses.

The main goal of inter-industry *balance is to find such a vector of a gross output* X, which with a known *matrix of direct expenses* A provides a given *vector of the final product* Y.

Vector X of gross output is found by the formula:

$$X = (E - A)^{-1} Y = S \cdot Y, \tag{5}$$

where the matrix $S = (E - A)^{-1}$ is called the matrix of *full expenses*, each element of which s_{ij} points out an amount of a gross output of i industry, which is necessary to ensure the release of the unit of the final product j industry $y_i = 1$ ($j = 1,2,...,n$).

Note 7.2 A matrix $A \geq 0$ is *productive*, if a solution $X \geq 0$ for equation (4) exists for any vector $Y \geq 0$. In this case linear model is called *productive*.

There are several criteria by which a matrix A is considered productive. So, if the matrix A is productive, then the maximum of the sums of elements of its column does not exceed one, at least for one of the columns the sum of elements is less than one. So, a matrix A is productive, if $a_{ij} \geq 0$ for any $i, j = 1,2,...,n$ and $\max\limits_{j=1,2,...,n} \sum\limits_{i=1}^{n} a_{ij} \leq 1$, and there exists number j such that $\sum\limits_{i=1}^{n} a_{ij} < 1$.

The net production of the branch is called the difference between the gross output of this branch and the expenses of production of all branches for the production of this branch.

We use the definitions above for solving some tasks.

Example 7.1 The coefficients of direct costs and final product of branches for the planned period are presented in the table 7.1 (in conventional currency):

Table 7.1

Branch		Consumption		Final product
		branch 1	branch 2	
Production	branch 1	0,35	0,25	350
	branch 2	0,27	0,14	280

Find:

1) planned about the gross production of goods, interim deliveries, net products of the goods;

2) the required volume of gross output of each industry, if the final consumption of products of the first industry will increase by 15 %, and the second industry will increase by 20 %.

Solution: 1) write down the direct expenses matrix A and the vector of final products:

$$A = \begin{pmatrix} 0{,}35 & 0{,}25 \\ 0{,}27 & 0{,}14 \end{pmatrix}, \quad Y = \begin{pmatrix} 350 \\ 280 \end{pmatrix}.$$

Notice that the matrix A is productive, because its elements are positive and the sum of the elements at the each row is less than one.

To find the direct expenses matrix, we find the matrix $E - A$:

$$E - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0{,}35 & 0{,}25 \\ 0{,}27 & 0{,}14 \end{pmatrix} = \begin{pmatrix} 0{,}65 & -0{,}25 \\ -0{,}27 & 0{,}86 \end{pmatrix}.$$

Hence, the matrix of full costs $S = (E - A)^{-1}$ is based on the previously considered algorithm for finding the inverse matrix:

$$\det(E - A) = \begin{vmatrix} 0{,}65 & -0{,}25 \\ -0{,}27 & 0{,}86 \end{vmatrix} = 0{,}559 - 0{,}0675 = 0{,}4915,$$

$$(E - A)^T = \begin{pmatrix} 0{,}65 & -0{,}27 \\ -0{,}25 & 0{,}86 \end{pmatrix},$$

$$A_{11}^T = 0{,}86, \quad A_{12}^T = 0{,}25, \quad A_{21}^T = 0{,}27, \quad A_{22}^T = 0{,}65,$$

$$S = (E - A)^{-1} = \frac{1}{0{,}4915} \begin{pmatrix} 0{,}86 & 0{,}25 \\ 0{,}27 & 0{,}65 \end{pmatrix} = \begin{pmatrix} 1{,}75 & 0{,}51 \\ 0{,}55 & 1{,}32 \end{pmatrix}.$$

Find the vector X of the gross product by the formula above:

$$X = S \cdot Y = \begin{pmatrix} 1{,}75 & 0{,}51 \\ 0{,}55 & 1{,}32 \end{pmatrix} \cdot \begin{pmatrix} 350 \\ 280 \end{pmatrix} = \begin{pmatrix} 612{,}5 + 142{,}8 \\ 192{,}5 + 369{,}6 \end{pmatrix} = \begin{pmatrix} 755{,}3 \\ 562{,}1 \end{pmatrix}.$$

The first row of the matrix X corresponds to branch 1, and the second row corresponds to branch 2.

We can find intersectoral supplies x_{ij} using the following formula:

$$x_{ij} = a_{ij}x_j,$$

$$x_{11} = a_{11} \cdot x_1 = 0{,}35 \cdot 755{,}3 = 264{,}35$$

$$x_{12} = a_{12} \cdot x_2 = 0{,}25 \cdot 562{,}1 = 140{,}53,$$

$$x_{21} = a_{21} \cdot x_1 = 0{,}27 \cdot 755{,}3 = 203{,}93$$

$$x_{22} = a_{22} \cdot x_2 = 0{,}14 \cdot 562{,}1 = 78{,}69.$$

The net production of the industry is equal to the difference between the gross output of this branch and the expenses of production of all branches for the production of this branch.

So, the production expenses of all branches are:

- for the first branch:

$$x_{11} + x_{21} = 264{,}35 + 203{,}93 = 468{,}28;$$

- for the second branch:

$$x_{12} + x_{22} = 140{,}53 + 78{,}69 = 219{,}22.$$

Finally, we have the following net products for

- the first branch: $755{,}3 - 468{,}28 = 287{,}02$;

- the second branch: $562{,}1 - 219{,}22 = 342{,}88$.

2) find the final consumption vector Y, taking into account that the final consumption of the first branch will increase by 15 %, and the second branch will increase by 20 %:

$$Y = \begin{pmatrix} 350 \cdot 1{,}15 \\ 280 \cdot 1{,}2 \end{pmatrix} = \begin{pmatrix} 402{,}5 \\ 336 \end{pmatrix}.$$

The latter makes it possible to find a vector X of gross output , which provides a given vector Y of the final product under a known matrix of direct expenses A:

$$X = S \cdot Y = \begin{pmatrix} 1,75 & 0,51 \\ 0,55 & 1,32 \end{pmatrix} \cdot \begin{pmatrix} 402,5 \\ 336 \end{pmatrix} = \begin{pmatrix} 704,38 + 171,36 \\ 221,38 + 443,52 \end{pmatrix} = \begin{pmatrix} 875,74 \\ 664,9 \end{pmatrix}.$$

Note 7.3 Data on total daily sales for the four stores is given in the table 7.2.

Table 7.2

Kind of products	store			
	№ 1	№ 2	№ 3	№ 4
product 1	35	51	36	70
product 2	30	46	35	61
product 3	50	49	52	48

The contents of this table can be presented in the form of a rectangular matrix which has a dimension 3×4:

$$A = \begin{pmatrix} 35 & 51 & 36 & 70 \\ 30 & 46 & 35 & 61 \\ 50 & 49 & 52 & 48 \end{pmatrix}.$$

Then it is easy to interpret every element of it. For example, the element a_{32} means, that the volume of the third product which have been sold in the second store is 49.

Example 7.2 Data on the aggregate supply of goods by some logistics company in the first and second quarters of the year are written by matrices:

$$A = \begin{pmatrix} 35 & 51 & 36 & 70 \\ 30 & 46 & 35 & 61 \\ 50 & 49 & 52 & 48 \end{pmatrix}, \quad B = \begin{pmatrix} 34 & 40 & 31 & 60 \\ 21 & 40 & 32 & 69 \\ 44 & 40 & 48 & 32 \end{pmatrix}.$$

Find data on the total volume of deliveries by logistics company for the first half of the year specified.

Solution. The requested data can be found by adding the given matrices and the result will be presented in this form:

$$C = A + B = \begin{pmatrix} 35 & 51 & 36 & 70 \\ 30 & 46 & 35 & 61 \\ 50 & 49 & 52 & 48 \end{pmatrix} + \begin{pmatrix} 34 & 40 & 31 & 60 \\ 21 & 40 & 32 & 69 \\ 44 & 40 & 48 & 32 \end{pmatrix} =$$

$$= \begin{pmatrix} 69 & 91 & 67 & 130 \\ 51 & 86 & 67 & 130 \\ 84 & 89 & 100 & 80 \end{pmatrix}.$$

Example 7.3 The store sells every day 45 pcs. of some goods for 1 UAH for a piece, 30 pcs. of some goods for 2 UAH for a piece and 50 pcs. of some goods for 0,5 UAH for a piece. Calculate the daily income from the sale of all goods.

Solution. We write down the given data about the sold goods as a matrix-row x, and the given data about the price of goods as a matrix column p:

$$x = \begin{pmatrix} 45 & 30 & 50 \end{pmatrix},\ p = \begin{pmatrix} 1 & 2 & 0,5 \end{pmatrix}.$$

Then the desired profit can be written as the product of matrices x and p^T. Note that p^T is a transpose matrix of the matrix p.

$$p^T = \begin{pmatrix} 1 \\ 2 \\ 0,5 \end{pmatrix},$$

$$S = x \cdot p^T = (45 \cdot 30 \cdot 50)\begin{pmatrix} 1 \\ 2 \\ 0,5 \end{pmatrix} = 45 + 30 \cdot 2 + 50 \cdot 0,5 = 130 .$$

Example 7.4 A trading company carries out retail, wholesaling of various goods, as well as sales through the Internet. Data on daily sales are recorded in the table 7.3:

Table 7.3

Sale	Product (price)		
	product 1 (1 thousand)	product 2 (2 thousand)	product 3 (0,5 thousand)
retail	45	30	50
wholesale	38	25	40
by Internet	20	15	20

Calculate daily profits from the sale of each type of service separately.

Solution. We write down data on daily sales as a matrix:

$$A = \begin{pmatrix} 45 & 30 & 50 \\ 38 & 25 & 40 \\ 20 & 15 & 20 \end{pmatrix},$$

and data on prices (in thousands of UAH) as a column matrix:

$$p = \begin{pmatrix} 1 \\ 2 \\ 0,5 \end{pmatrix}.$$

The desired daily incomes (or profits) u_1, u_2, u_3 from the sale of each of the three types of goods can be written as a matrix column u and defined as the product of matrices A and p:

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = A\,p = \begin{pmatrix} 45 & 30 & 50 \\ 38 & 25 & 40 \\ 20 & 15 & 20 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0,5 \end{pmatrix} =$$

$$= \begin{pmatrix} 45\cdot 1 + 30\cdot 2 + 50\cdot 0,5 \\ 38\cdot 1 + 25\cdot 2 + 40\cdot 0,5 \\ 20\cdot 1 + 15\cdot 2 + 20\cdot 0,5 \end{pmatrix} = \begin{pmatrix} 130 \\ 108 \\ 60 \end{pmatrix}.$$

Example 7.5 The company produces three kinds of products: pies, cakes and brownies; and uses three types of raw materials S_1, S_2, S_3. The raw material consumption rates for one type of product and the volume of their consumption for one day for production of each product are presented in the table 7.4:

Table 7.4

Type of raw materials	Raw material consumption rates for one type of product, standard units			The volume of raw material consumption for one day, standard units
	pie	cake	brownie	
S_1	5	3	4	2700
S_2	2	1	1	800
S_3	3	2	2	1600

Find the daily output of each type of products.

Solution. Let the company produces x_1 pies, x_2 cakes, x_3 brownies. Then according to the raw material costs of each type we have a system:

$$\begin{cases} 5x_1 + 3x_2 + 4x_3 = 2700; \\ 2x_1 + x_2 + x_3 = 800; \\ 3x_1 + 2x_2 + 2x_3 = 1600. \end{cases}$$

(The system should be solved by yourself)

When we solved this system, we get $x_1 = 200$, $x_2 = 300$, $x_3 = 200$, that is: the company produce 200 pies, 300 cakes and 200 brownies.

Example 7.6 The manufacturing company offers three types of goods using four types of resources. Resource consumption rates of i goods to provide the production unit of j kind of goods is given at the matrix A. Suppose that for a certain period of time the enterprise will offer the quantity of each type of goods, which is given by the matrix X, and the cost of each type of resources per unit is presented in the form of a matrix P:

$$A = \begin{pmatrix} 4 & 7 & 9 \\ 8 & 4 & 3 \\ 6 & 2 & 8 \\ 5 & 4 & 9 \end{pmatrix}, \quad X = \begin{pmatrix} 275 \\ 230 \\ 495 \end{pmatrix}, \quad P = \begin{pmatrix} 160 & 10 & 25 & 45 \end{pmatrix}.$$

Find:

a) a matrix S of the total expenses of resources of each type for the preparation of all goods for a certain period;

б) a matrix C of the total cost of all spent resources for a certain period.

Solution: a) a matrix of the total expenses of resources of each type for the preparation of all goods for a certain period can be found by formula $S = A \cdot X$:

$$S=\begin{pmatrix}4 & 7 & 9\\ 8 & 4 & 3\\ 6 & 2 & 8\\ 5 & 4 & 9\end{pmatrix}\cdot\begin{pmatrix}275\\ 230\\ 495\end{pmatrix}=\begin{pmatrix}4\cdot 275+7\cdot 230+9\cdot 495\\ 8\cdot 275+4\cdot 230+3\cdot 495\\ 6\cdot 275+2\cdot 230+8\cdot 495\\ 5\cdot 275+4\cdot 230+9\cdot 495\end{pmatrix}=$$

$$=\begin{pmatrix}1100+1610+4455\\ 2200+920+1485\\ 1650+460+3960\\ 1375+920+4455\end{pmatrix}=\begin{pmatrix}7165\\ 4605\\ 6070\\ 6750\end{pmatrix};$$

б) a matrix of the total cost of all spent resources for a certain period can be found due to using formulas $C=P\cdot A\cdot X$ or $C=P\cdot S$:

$$C=\begin{pmatrix}160 & 10 & 25 & 45\end{pmatrix}\cdot\begin{pmatrix}7165\\ 4605\\ 6070\\ 6750\end{pmatrix}=1647950.$$

Consequently, the total cost of the spent resources is 1647950 of monetary units.

Questions for self-control

1. In what field of research are matrices applied?
2. What is an input–output model?
3. What could be analyzed by Leontief's model?
4. How can we know if consumption matrix is productive or not?
5. What is resulting demand? How we can find it?
6. What is an amount of production vector?
7. Are systems used to perform economic analysis? Give some examples.

Tasks for revision

7.1 The direct costs and final product of branches for the planned period (in conventional monetary units) are given in the table 7.5:

Table 7.5

Branch		Consumption		final product
		branch 1	branch 2	
Production	branch 1	0,25	0,4	200
	branch 2	0,15	0,3	400

Find: 1) planned gross production of goods, interim deliveries, net products of the goods; 2) the required volume of gross output of each branch, if the final consumption of products of the first industry will increase by 10 %, and if the final consumption of products of the second industry will increase by 20 %.

7.2 Given matrix S of full costs of some model of inter-industry balance. Find: a) increase in gross output ΔX_1, which would ensure the growth of the final product ΔY_1; б) increase in the final product ΔY_2, which corresponds to the growth of gross output ΔX_2:

$$S = \begin{pmatrix} 0,5 & 1,2 & 0,3 \\ 0,4 & 0,2 & 1,1 \\ 0,7 & 0,6 & 0,2 \end{pmatrix}, \ \Delta Y_1 = \begin{pmatrix} 20 \\ 10 \\ 30 \end{pmatrix}, \ \Delta X_2 = \begin{pmatrix} -5 \\ 20 \\ 15 \end{pmatrix}.$$

8 The homogeneous square systems of linear algebraic equations

As previously stated (look at page 3) the homogeneous rectangular SLAE, having the matrix equation as $AX = 0$, is always compatible and has a trivial (zero) solution, that is $X = 0$. This statement is true because, according to Kroneker-Kappelli theorem, the rank of the augmented matrix $\widetilde{A} = (A|0)$ equals the rank of the coefficients matrix A. The zero solution $X = 0$ will be unique if their common rank of matrices equals the number of the system unknowns. Otherwise, the homogeneous SLAE will have an infinite set of solutions.

Therefore, we can formulate the following theorem for the square system of linear equations.

Theorem 8.1 *The homogeneous square system* $AX = 0$ *has a non-zero solution then and only if the determinant of a system equals zero,* $\det A = 0$. *In the case if the determinant is different from zero, a system has only a zero solution.*

Example 8.1 Find a value of the parameter α, for which the homogeneous square system

$$\begin{cases} x_1 - \alpha x_2 + 7x_3 = 0 \\ x_1 + 3x_2 - 5x_3 = 0 \\ x_1 - x_2 + \alpha x_3 = 0 \end{cases}$$

has a non-zero solution (or an infinite set of solutions).

Solution: As we know, the given system can have a non-zero solution (according to the theorem 8.1), if the main determinant equals zero, $\Delta = \det A = 0$, so we write down this determinant and equate it to zero. We solve the equation:

$$\begin{vmatrix} 1 & -\alpha & 7 \\ 1 & 3 & -5 \\ 1 & -1 & \alpha \end{vmatrix} = 0; \quad \alpha^2 + 8\alpha - 33 = 0 \; ; \quad \alpha_1 = 3; \quad \alpha_2 = -11.$$

Answer: This system will have a non-zero solution if value of parameter is $\alpha = 3$ or $\alpha = -11$.

Example 8.2 Make sure that given homogeneous square system has an infinite set of solutions. Find its general solution and any non-zero partial solution:

$$\begin{cases} 2x_1 - x_2 + 3x_3 = 0; \\ x_1 - 3x_2 + 2x_3 = 0; \\ x_1 + 2x_2 + x_3 = 0. \end{cases}$$

Solution: $\Delta = \det A = \begin{vmatrix} 2 & -1 & 3 \\ 1 & -3 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 0$.

Hereby, our system has an infinite set of solutions. We solve this system by Gaussian elimination method.

The straight course: Swap the first row and the second row

$$\widetilde{A} = (A|0) = \left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 2 & -1 & 3 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right) \sim$$

multiply the first row by (–2) and add it to the second row; subtract the first row from the third row and the second row from the third row. In the result we have a matrix

$$\sim \left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 5 & -1 & 0 \\ 0 & 5 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 5 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

As we can see, after performing the row operations on the augmented matrix of a homogeneous system of linear equations the last column of the matrix is all zeros. Any one of three allowable row operations will convert zeros to zeros and thus, the final column of the matrix in reduced row-echelon form will also be all zeros. So in this case, we may likely refer only to the coefficient matrix because we remember that the final column consists of zeros, and after any number of row operations have been performed, is still zero.

Thus,

$$rang\,\widetilde{A} = rang\,A = r = 2 < n = 3\,,$$

There are x_1, x_2 which are the basic unknowns; x_3 is a free unknown.

<u>The reverse course</u>. From the last row we get $0x_3 = 0$. As we know that the solution of such an equation is any value of x_3, so free unknown can be accepted for an arbitrary constant (parameter), namely: $x_3 = t$, $t \in R$. We compose the system of equations from the obtained matrix, without taking into consideration of the zero-row, and solve it.

$$\begin{cases} x_1 - 2x_2 + 2x_3 = 0; \\ \qquad 5x_2 - x_3 = 0. \end{cases}$$

Remove all elements which have a free unknown x_3 to the right. The obtained system has an upper triangular form relative to principal unknowns x_1, x_2:

$$\begin{cases} x_1 - 2x_2 = -2x_3; \\ \qquad x_2 = \dfrac{1}{5}x_3. \end{cases}$$

Solve the given system, starting from the last equation.

$$x_3 = t\,;\;\; x_2 = \frac{1}{5}t\,;\; x_1 = 2x_2 - 2t = 2\cdot\frac{1}{5}t - 2t = -\frac{8}{5}t\,.$$

Thus, the general solution is

$$x_1 = -\frac{8}{5}t\,;\quad x_2 = \frac{1}{5}t\,;\quad x_3 = t\,,\quad t \in R\,.$$

Let us $t=-5$. Then we have a non-zero particular solution as: $x_1=8$; $x_2=-1$; $x_3=-5$.

Answer: $x_1=-\frac{8}{5}t$, $x_2=\frac{1}{5}t$, $x_3=t$, $t\in R$ is the general solution; $x_1=8$, $x_2=-1$, $x_3=-5$ is the particular solution.

Example 8.3 Solve the system

$$\begin{cases} x_1-4x_2+7x_3=0; \\ x_1+3x_2-5x_3=0; \\ 2x_1-x_2+x_3=0. \end{cases}$$

Solution. Calculate the main determinant of system

$$\begin{vmatrix} 1 & -4 & 7 \\ 1 & 3 & -5 \\ 2 & -1 & 1 \end{vmatrix}=3+40-7-42+4-5=-7.$$

According to theorem 8.1, this system has only one unique solution as zero-solution: $x_1=x_2=x_3=0$.

Answer: $x_1=x_2=x_3=0$.

Questions for self-control

1. What is a homogeneous system?
2. How many solutions can a homogeneous system have?
3. Formulate the condition for the presence of non-zero solutions in a square SLAE.
4. How are block matrices used to solve SLAE and find the inverse matrix?
5. What should you do to find the non-zero solution of a homogeneous system?

Tasks for revision

8.1 Solve the system $\begin{cases} x_1+5x_2+3x_3=0; \\ x_1+3x_2+5x_3=0; \\ 2x_1+x_2+x_3=0. \end{cases}$

8.2 Solve the system $\begin{cases} x_1+5x_2+3x_3=0; \\ x_1+3x_2+5x_3=0; \\ 2x_1+10x_2+6x_3=0. \end{cases}$

8.3 Solve the system $\begin{cases} x_1+5x_2+3x_3=0; \\ x_1+3x_2+5x_3+x_4=0; \\ 2x_1+10x_2+6x_3+x_4=0; \\ -x_1-5x_2-3x_3=0. \end{cases}$

9 Eigenvalues and eigenvectors of the matrix

Let n be an arbitrary fixed natural number. Any ordered set of n real numbers $(x_1, x_2, \dots, x_n)$ is called the *n-dimensional point M*, so $M(x_1, x_2, \dots, x_n)$. Set of n-dimensional points is called *n-dimensional point space R^n*. Numbers $x_1, x_2, \dots, x_n$ are *coordinates* of the point M. The number n is called a *space dimension*.

Let A be a square matrix having dimension n by n. Consider the corresponding linear mapping of a space R^n by itself: $\vec{y}=A\vec{x}$. If the non-zero vector $\vec{x}$ and a number λ are such at which the equality $A\vec{x}=\lambda\vec{x}$ is performed, then one says that λ is *the eigenvalue* of a matrix A, and $\vec{x}$ is *the eigenvector* corresponding to the eigenvalue λ.

Consequently, the multiplication of the matrix by its eigenvector is equivalent to multiplication of the eigenvalue with this vector. The mentioned matrix equation can be presented in the form:

$$A\vec{x} = \lambda E\vec{x}\,,\ (A-\lambda E)\vec{x} = \vec{0}\,.$$

This homogeneous square system of linear equations has non-zero solution $\vec{x}$ then and only then if its determinant equals zero:

$$\det(A-\lambda E) = 0\,.$$

This equation is called *a characteristic equation of a matrix* A. The corresponding polynomial

$$f(\lambda) = \det(A-\lambda E)$$

is called *a characteristic polynomial of a matrix* A.

The characteristic equation can be written in expanded form as:

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0\,.$$

The eigenvalues λ_j $(j=\overline{1,n})$ are the roots of characteristic equation.

Set of all eigenvalues λ_j $(j=\overline{1,n})$ of the given matrix is called its *spectrum*.

If some eigenvalue λ is known, then the corresponding eigenvector can be found from the homogeneous system $(A-\lambda E)\vec{x} = \vec{0}$.

Properties of eigenvalues:

1. Each eigenvector corresponds to the one eigenvalue.
2. If $\vec{x}$ is an eigenvector and λ is its eigenvalue, then an

arbitrary vector $\alpha\vec{x}$, $\alpha \neq 0$, which is a collinear vector to the vector $\vec{x}$, and also it is an eigenvector with the same eigenvalue λ. Thus, a eigenvector is defined up to an arbitrary non-zero multiplier. Usually unit eigenvectors are isolated.

3. If $\vec{x}_1$ and $\vec{x}_2$ are eigenvectors of matrix A which have the same eigenvalue as λ, then their sum $\vec{x}_1 + \vec{x}_2$ is eigenvector of matrix A and it has the same eigenvalue as λ.

4. The determinant of the matrix A equals to the product of all its eigenvalues:

$$\det A = \prod_{j=1}^{n} \lambda_j = \lambda_1 \lambda_2 \ldots \lambda_n .$$

5. *Trace* of an n-by-n square matrix A is defined as a sum of the elements of the main diagonal $SpA = \sum_{j=1}^{n} a_{ij} = a_{11} + a_{22} + \ldots + a_{nn}$. Trace of the matrix A equals to the sum of all its eigenvalues

$$SpA = \sum_{j=1}^{n} \lambda_j = \lambda_1 + \lambda_2 + \ldots + \lambda_n .$$

Example 9.1 Find eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Solution. We compose and solve the characteristic equation:

$$|A - \lambda E| = 0,$$

$$A - \lambda E = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} =$$

$$=\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix},$$

$$|A-\lambda E| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\cdot(1-\lambda)-4 = 1-2\lambda+\lambda^2-4 =$$

$$=\lambda^2-2\lambda-3,\quad \lambda^2-2\lambda-3=0,$$

$$D=(-2)^2-4\cdot1\cdot(-3)=4+12=16,$$

$$\lambda_1=\frac{-(-2)+\sqrt{16}}{2\cdot1}=\frac{2+4}{2}=3,$$

$$\lambda_2=\frac{-(-2)-\sqrt{16}}{2\cdot1}=\frac{2-4}{2}=\frac{-2}{2}=-1.$$

Thus, our values $\lambda_1=3$, $\lambda_2=-1$ are the eigenvalues.

Find the corresponding eigenvectors.

Let $\lambda_1=3$, we substitute it into a homogeneous system of equations:

$$(A-\lambda E)\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0,$$

$$\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0,\quad \begin{pmatrix} 1-3 & 2 \\ 2 & 1-3 \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0,$$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0.$$

We will solve the system of equations: $\begin{cases} -2x_1+2x_2=0; \\ 2x_1-2x_2=0. \end{cases}$

We get from these two equations that $x_1 = x_2$. Let $x_2 = 1$, then $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thereby, the eigenvalue $\lambda_1 = 3$ corresponds to the eigenvector $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let $\lambda_2 = -1$, we substitute it into a homogeneous system of equations:

$$(A - \lambda E) \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0, \quad \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0,$$

$$\begin{pmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0, \quad \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

We will solve the system of equations: $\begin{cases} 2x_1 + 2x_2 = 0; \\ 2x_1 + 2x_2 = 0. \end{cases}$

From those two equations we get: $x_1 = -x_2$. Let $x_2 = 1$, then $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Thereby, the eigenvalue $\lambda_2 = -1$ corresponds to the eigenvector $X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Answer: $\lambda_1 = 3$, $\lambda_2 = -1$, $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Example 9.2 Find eigenvalues and eigenvectors of the matrix

$$A=\begin{pmatrix}1 & 1 & -1\\ 1 & 1 & 1\\ -1 & 1 & -1\end{pmatrix}.$$

Solution. We compose and solve the characteristic equation:

$$|A-\lambda E|=0,$$

$$A-\lambda E=\begin{pmatrix}1 & 1 & -1\\ 1 & 1 & 1\\ -1 & 1 & -1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 1 & -1\\ 1 & 1 & 1\\ -1 & 1 & -1\end{pmatrix}-\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}=$$

$$=\begin{pmatrix}1-\lambda & 1 & -1\\ 1 & 1-\lambda & 1\\ -1 & 1 & -1-\lambda\end{pmatrix},$$

after this we calculate the determinant of the obtained matrix as:

$$\det\left(A-\lambda E\right)=\begin{vmatrix}1-\lambda & 1 & -1\\ 1 & 1-\lambda & 1\\ -1 & 1 & -1-\lambda\end{vmatrix}=0,$$

$$(1-\lambda)\cdot\begin{vmatrix}1-\lambda & 1\\ 1 & -1-\lambda\end{vmatrix}-1\cdot\begin{vmatrix}1 & 1\\ -1 & -1-\lambda\end{vmatrix}-1\cdot\begin{vmatrix}1 & 1-\lambda\\ -1 & 1\end{vmatrix}=0,$$

$$\left(1-\lambda\right)\left(\left(1-\lambda\right)\left(-1-\lambda\right)-1\right)-(-1-\lambda+1)-(1+1-\lambda)=0,$$

$$\left(1-\lambda\right)\left(\lambda^2-1-1\right)+\lambda-2+\lambda=0,$$

$$\left(1-\lambda\right)\left(\lambda^2-2\right)-2\left(1-\lambda\right)=0,$$

$$(1-\lambda)\left(\lambda^2-4\right)=0,$$

solving this equation, we get:

$$\lambda_1=1,\quad \lambda_{2,3}=\pm 2.$$

Thus, the eigenvalues are $\lambda_1=1$, $\lambda_{2,3}=\pm 2$.

Further, we find the corresponding eigenvector.

Let $\lambda_1=1$, we substitute it into a homogeneous system of equations:

$$(A-\lambda E)\cdot\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0,$$

$$\begin{pmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ -1 & 1 & -1-\lambda \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0,$$

$$\begin{pmatrix} 1-1 & 1 & -1 \\ 1 & 1-1 & 1 \\ -1 & 1 & -1-1 \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0,$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & -2 \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0.$$

We will solve the system of equations: $\begin{cases} 0x_1+x_2-x_3=0; \\ x_1+0x_2+x_3=0; \\ -x_1+x_2-2x_3=0. \end{cases}$

We have: $x_2 = x_3$, $x_1 = -x_3$, $x_3 \in R$. Let $x_3 = 1$, then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. Thereby, the eigenvalue $\lambda_1 = 1$ corresponds to the eigenvector $X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

Let $\lambda_2 = 2$, we substitute it into a homogeneous system of equations:

$$(A - \lambda E) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0, \quad \begin{pmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ -1 & 1 & -1-\lambda \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

$$\begin{pmatrix} 1-2 & 1 & -1 \\ 1 & 1-2 & 1 \\ -1 & 1 & -1-2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0, \quad \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

We will solve the system of equations:

$$\begin{cases} -x_1 + x_2 - x_3 = 0; \\ x_1 - x_2 + x_3 = 0; \\ -x_1 + x_2 - 3x_3 = 0. \end{cases}$$

We have: $$\begin{cases} x_1 - x_2 + x_3 = 0; \\ x_2 - 2x_3 = 0; \\ 0 \cdot x_3 = 0, \end{cases} \quad \begin{cases} x_3 \in R; \\ x_2 = 2x_3; \\ x_1 = x_3. \end{cases}$$

Let $x_3 = 1$, then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. Thereby, the eigenvalue $\lambda_2 = 2$ corresponds to the eigenvector $X_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Let $\lambda_3 = -2$, we substitute it into a homogeneous system of equations:

$$(A - \lambda E) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

$$\begin{pmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ -1 & 1 & -1-\lambda \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

$$\begin{pmatrix} 1+2 & 1 & -1 \\ 1 & 1+2 & 1 \\ -1 & 1 & -1+2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0, \quad \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

We will solve the system of equations:

$$\begin{cases} 3x_1 + x_2 - x_3 = 0; \\ x_1 + 3x_2 + x_3 = 0; \\ -x_1 + x_2 + x_3 = 0. \end{cases}$$

We have: $\begin{cases} x_1 + 3x_2 + x_3 = 0; \\ 2x_2 + x_3 = 0; \\ 0 \cdot x_3 = 0, \end{cases}$ $\begin{cases} x_3 \in R; \\ x_2 = -\dfrac{1}{2} x_3; \\ x_1 = \dfrac{1}{2} x_3. \end{cases}$

Let $x_3 = 1$, then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$. Thereby, the eigenvalue

$\lambda_3 = -2$ corresponds to the eigenvector $X_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$.

Answer: $\lambda_1 = 1$, $\lambda_3 = -2$, $\lambda_2 = 2$, $X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$,

$$X_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},\ X_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}.$$

Questions for self-control

1. What is the characteristic equation? What can we get from the characteristic equation?
2. What is a root of the characteristic equation?

3. What are the eigenvalues and eigenvectors of the square matrix?

4. What do we name the eigenvalues and eigenvectors?

5. Formulate the properties of the eigenvalues and eigenvectors.

6. What are differences between eigenvalues and eigenvectors?

Tasks for revision

9.1 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 100 \\ 100 & 1 \end{pmatrix}.$$

9.2 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}.$$

9.3 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}.$$

10 Matrix polynomials

Let A be a square matrix with an arbitrary dimension n. If at the arbitrary polynomial

$$f(x) = a_0 x^m + a_1 x^{m-1} + \ldots + a_{m-1} x + a_m$$

we substitute a matrix A instead of x, then we get

$$f(A) = a_0 A^m + a_1 A^{m-1} + \ldots + a_{m-1} A + a_m E \,,$$

which is called as *polynomial of the matrix* A *(the matrix polynomial).*

Note 10.1 Algebraic actions can be carried out on matrix polynomials as well as on ordinary polynomials.

Cayley-Hamilton theorem. An arbitrary square matrix is a root of its characteristic matrix polynomial.

(Without proof).

Example 10.1 Check out that the given matrix

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$$

is a root of its characteristic matrix polynomial?

Solution. Find the characteristic matrix polynomial:

$$f(\lambda) = \det(A - \lambda E) = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10 \,.$$

Calculate $f(A)$:

$$f(A) = A^2 - 3A - 10E = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}^2 - 3\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} - 10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1\cdot 1 + 3\cdot 4 & 1\cdot 3 + 3\cdot 2 \\ 4\cdot 1 + 2\cdot 4 & 4\cdot 3 + 2\cdot 2 \end{pmatrix} - \begin{pmatrix} 3 & 9 \\ 12 & 6 \end{pmatrix} - \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} =$$

$$= \begin{pmatrix} 13-3-10 & 9-9-0 \\ 12-12-0 & 16-6-10 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Questions for self-control

1. What is a matrix polynomial?
2. What arithmetic operations can we do with matrix polynomial?
3. Formulate the Cayley-Hamilton theorem about a characteristic polynomial.
4. What should you do to check out if the matrix is a root of the characteristic equation?
5. How can we solve SLAE by the method of simple iterations?

Tasks for revision

10.1 Find the value of the polynomial $f(x) = 3x^2 - 5x + 2$ of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$.

10.2 Find the value of the matrix polynomial $f(A) = 2A^2 + 3A + 5E$, if $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 4 & 1 & 1 \end{pmatrix}$, and E is identity matrix of the third order.

10.3 Find the value of the matrix polynomial $f(A) = 7A - E$, if $A = \begin{pmatrix} a & 0 & 0 \\ a & a & 0 \\ a & a & a \end{pmatrix}$, and E is identity matrix of the third order.

11 Application the systems of linear algebraic equations in chemistry

In chemistry, when solving certain problems, mathematical methods are used. In particular, the solution of systems of linear algebraic equations is used, for example, to analyze mixtures and calculate the equilibrium of multicomponent systems. Consider the examples.

Example 11.1 Let us mix, under controlled conditions, toluene C_7H_8 and nitric acid HNO_3 to produce trinitrotoluene $C_7H_5O_6N_3$ along with the byproduct water. In what proportions should they be mixed?

Solution. The number of atoms of element before the reaction must equal the number of its atoms after the reaction:

$$x\,C_7H_8 + y\,HNO_3 \to z\,C_7H_5O_6N_3 + t\,H_2O\,.$$

Applying that to the elements C, H, N, and O gives the following system:

$$\begin{cases} 7x = 7z; \\ 8x + y = 5z + 2t; \\ y = 3z; \\ 3y = 6z + t, \end{cases} \quad \text{or} \quad \begin{cases} 7x - 7z = 0; \\ 8x + y - 5z - 2t = 0; \\ y - 3z = 0; \\ 3y - 6z - t = 0. \end{cases}$$

We obtained a homogeneous system of linear algebraic equations. Let us solve it by the Gauss method. An augmented matrix of the system has the form:

$$\begin{pmatrix} 7 & 0 & -7 & 0 \\ 8 & 1 & -5 & -2 \\ 0 & 1 & -3 & 0 \\ 0 & 3 & -6 & -1 \end{pmatrix}.$$

Let us divide the first row by 7. Then we multiply the first row by (-8) and add to the second one:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 8 & 1 & -5 & -2 \\ 0 & 1 & -3 & 0 \\ 0 & 3 & -6 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & -3 & 0 \\ 0 & 3 & -6 & -1 \end{pmatrix} \sim$$

Multiply the second row by (-1) and add to the third one, then by (-3) and add to the fourth row:

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -15 & 5 \end{pmatrix} \sim$$

Divide the fourth row by (-5). Then we will interchange the third and fourth rows. Further, we multiply the third row by 2 and add to the fourth one:

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -6 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

As a result of transformations, the matrix has a trapezium form, which means the system is indefinite, consistent and has many solutions.

The system can be written in the form:

$$\begin{cases} x - z = 0; \\ y + 3z - 2t = 0; \\ 3z - t = 0. \end{cases}$$

Let $t = k$, where $k \in R$. Remove the members with a free unknown $t = k$ to the right side. We obtain the system of the upper triangular form with respect to the basic unknowns x, y, z and solve it, rising from the bottom up:

$$\begin{cases} x - z = 0; \\ y + 3z = 2k; \\ 3z = k, \end{cases} \quad \begin{cases} x - \dfrac{k}{3} = 0; \\ y + 3 \cdot \dfrac{k}{3} = 2k; \\ z = \dfrac{k}{3}, \end{cases} \quad \begin{cases} x = \dfrac{k}{3}; \\ y + k = 2k; \\ z = \dfrac{k}{3}, \end{cases} \quad \begin{cases} x = \dfrac{k}{3}; \\ y = k; \\ z = \dfrac{k}{3}. \end{cases}$$

That is, the general solution has the form:

$$x = \frac{k}{3},\ y = k,\ z = \frac{k}{3},\ t = k,\ k \in R.$$

Let us check the obtained solution:

$$\begin{cases} 7 \cdot \dfrac{k}{3} - 7 \cdot \dfrac{k}{3} = 0; \\ 8 \cdot \dfrac{k}{3} + k - 5 \cdot \dfrac{k}{3} - 2k = k + k - 2k = 0; \\ k - 3 \cdot \dfrac{k}{3} = k - k = 0; \\ 3k - 6 \cdot \dfrac{k}{3} - k = 3k - 2k - k = 0. \end{cases}$$

The solution set has many vectors besides the zero vector (if we take k to be a number of molecules then solution makes sense only when k is a nonnegative multiple of 3).

Answer: $x=\frac{k}{3}$, $y=k$, $z=\frac{k}{3}$, $t=k$, $k\in R$.

Example 11.2 When dissolved in acid, 2.33 g of a mixture of iron and zinc, 896 mL of hydrogen were obtained. How many grams of each metal were in the mixture?

Solution. The task says about the interaction of a mixture of metals with acid. So, at the same time there are two reactions: zinc with acid and iron with acid. Herewith, the corresponding salts are formed, and hydrogen is emitted, the total volume of which is 896 L:

$$\mathrm{Fe+2HCl\rightarrow FeCl+H_2},\quad \mathrm{Zn+2HCl\rightarrow ZnCl_2+H_2}.$$

First way. Let it be the mixture x g of iron and y g of zinc:

$$\underset{56\,\mathrm{g}}{\overset{x\,\mathrm{g}}{\mathrm{Fe}}}+\mathrm{2HCl\rightarrow FeCl}+\underset{22{,}4\ \mathrm{L}}{\overset{y_1\,\mathrm{L}}{\mathrm{H_2}}},\quad \underset{65\,\mathrm{g}}{\overset{y\,\mathrm{g}}{\mathrm{Zn}}}+\mathrm{2HCl\rightarrow ZnCl_2}+\underset{22{,}4\ \mathrm{L}}{\overset{y_2\,\mathrm{L}}{\mathrm{H_2}}}.$$

According to the condition of the problem:

$$\begin{cases} x+y=2{,}33; \\ y_1+y_2=0{,}896, \end{cases}$$

$$\begin{cases} x+y=2{,}33; \\ \dfrac{22{,}4x}{56}+\dfrac{22{,}4y}{65}=0{,}896, \end{cases}\qquad \begin{cases} x+y=2{,}33; \\ 0{,}4x+0{,}345y=0{,}896. \end{cases}$$

Let us solve the system by the Cramer's rule:

$$\Delta=\begin{vmatrix} 1 & 1 \\ 0{,}4 & 0{,}345 \end{vmatrix}=0{,}345-0{,}4=-0{,}055,$$

$$\Delta_1 = \begin{vmatrix} 2,33 & 1 \\ 0,896 & 0,345 \end{vmatrix} = 0,80385 - 0,896 = -0,09215,$$

$$\Delta_2 = \begin{vmatrix} 1 & 2,33 \\ 0,4 & 0,896 \end{vmatrix} = 0,896 - 0,932 = -0,036,$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-0,09215}{-0,055} = 1,68, \quad y = \frac{\Delta_2}{\Delta} = \frac{-0,036}{-0,055} = 0,65.$$

Second way. Let in a mixture x moles of iron and y moles of zinc be, then $m(\text{Fe}) = 56x$, and $m(\text{Zn}) = 65y$. And the system of equations will take the form:

$$\begin{cases} 56x + 65y = 2,33; \\ 22,4x + 22,4y = 0,896. \end{cases}$$

Let us solve the system by the Cramer's rule:

$$\Delta = \begin{vmatrix} 56 & 65 \\ 22,4 & 22,4 \end{vmatrix} = 1254,4 - 1456 = -201,6,$$

$$\Delta_1 = \begin{vmatrix} 2,33 & 65 \\ 0,896 & 22,4 \end{vmatrix} = 52,192 - 58,24 = -6,048,$$

$$\Delta_2 = \begin{vmatrix} 56 & 2,33 \\ 22,4 & 0,896 \end{vmatrix} = 50,176 - 52,192 = -2,016,$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-6,048}{-201,6} = 0,03, \quad y = \frac{\Delta_2}{\Delta} = \frac{-2,016}{-201,6} = 0,01.$$

Then

$m(\text{Fe}) = 56 \cdot 0,03 = 1,68$ g, a $m(\text{Zn}) = 65 \cdot 0,01 = 0,65$ g.

Answer: $m(\text{Fe}) = 1,68$ g, $m(\text{Zn}) = 0,65$ g.

Example 11.3 Four tanks with solutions of sulfuric acid of different concentrations are given. Basic data are presented in the table 11.1. If one mixes the solutions in certain ratios, an acid of a given concentration will be received. Determine the concentration of acid in each vessel.

Table 11.1

Concentration ratio	Final acid concentration, %
1:1:1:1	13
4:3:2:1	34
4:1:1:4	25
4:1:4:1	25

Solution. Let x_1, x_2, x_3, x_4 be unknown concentrations of solutions of sulfuric acid in four tanks. Then

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 13; \\ 4x_1 + 3x_2 + 2x_3 + x_4 = 34; \\ 4x_1 + x_2 + x_3 + 4x_4 = 25; \\ 4x_1 + x_2 + 4x_3 + x_4 = 25. \end{cases}$$

Let us solve the system by Gaussian elimination method. An augmented matrix of the system has the form:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 13 \\ 4 & 3 & 2 & 1 & 34 \\ 4 & 1 & 1 & 4 & 25 \\ 4 & 1 & 4 & 1 & 25 \end{array}\right).$$

Multiply the first row by (-4) and add to the second, third and fourth ones. Then we multiply the second row by (-3) and add to the third and fourth ones. Further, we multiply the third row by (-2) and add to the fourth one:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 13 \\ 0 & -1 & -2 & -3 & -18 \\ 0 & -3 & -3 & 0 & -27 \\ 0 & -3 & 0 & -3 & -27 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 13 \\ 0 & -1 & -2 & -3 & -18 \\ 0 & 0 & 3 & 9 & 27 \\ 0 & 0 & 6 & 6 & 27 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 13 \\ 0 & -1 & -2 & -3 & -18 \\ 0 & 0 & 3 & 9 & 27 \\ 0 & 0 & 0 & -12 & -27 \end{array}\right).$$

The resulting matrix has a triangular form, therefore, the system will have a unique solution:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 13; \\ -x_2 - 2x_3 - 3x_4 = -18; \\ 3x_3 + 9x_4 = 27; \\ -12x_4 = -27, \end{cases} \qquad \begin{cases} x_1 + x_2 + x_3 + x_4 = 13; \\ -x_2 - 2x_3 - 3x_4 = -18; \\ 3x_3 + 9 \cdot \dfrac{9}{4} = 27; \\ x_4 = \dfrac{-27}{-12} = \dfrac{9}{4}, \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 13; \\ -x_2 - 2x_3 - 3x_4 = -18; \\ 3x_3 = 27 - \dfrac{81}{4} = \dfrac{27}{4}; \\ x_4 = \dfrac{9}{4}, \end{cases} \qquad \begin{cases} x_1 + x_2 + x_3 + x_4 = 13; \\ -x_2 - 2 \cdot \dfrac{9}{4} - 3 \cdot \dfrac{9}{4} = -18; \\ x_3 = \dfrac{9}{4}; \\ x_4 = \dfrac{9}{4}, \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 13; \\ -x_2 = -18 + \dfrac{18}{4} + \dfrac{27}{4} = \dfrac{-72+18+27}{4} = -\dfrac{27}{4}; \\ x_3 = \dfrac{9}{4}; \\ x_4 = \dfrac{9}{4}, \end{cases}$$

$$\begin{cases} x_1 + \dfrac{27}{4} + \dfrac{9}{4} + \dfrac{9}{4} = 13; \\ x_2 = \dfrac{27}{4}; \\ x_3 = \dfrac{9}{4}; \\ x_4 = \dfrac{9}{4}, \end{cases} \quad \begin{cases} x_1 = 13 - \dfrac{27}{4} - \dfrac{9}{4} - \dfrac{9}{4} = \dfrac{52-45}{4} = \dfrac{7}{4}; \\ x_2 = \dfrac{27}{4}; \\ x_3 = \dfrac{9}{4}; \\ x_4 = \dfrac{9}{4}. \end{cases}$$

Let us check out the obtained solution:

$$\begin{cases} \dfrac{7}{4} + \dfrac{27}{4} + \dfrac{9}{4} + \dfrac{9}{4} = \dfrac{52}{4} = 13; \\ 4 \cdot \dfrac{7}{4} + 3 \cdot \dfrac{27}{4} + 2 \cdot \dfrac{9}{4} + \dfrac{9}{4} = \dfrac{28}{4} + \dfrac{81}{4} + \dfrac{18}{4} + \dfrac{9}{4} = \dfrac{136}{4} = 34; \\ 4 \cdot \dfrac{7}{4} + \dfrac{27}{4} + \dfrac{9}{4} + 4 \cdot \dfrac{9}{4} = \dfrac{28}{4} + \dfrac{27}{4} + \dfrac{9}{4} + \dfrac{36}{4} = \dfrac{100}{4} = 25; \\ 4 \cdot \dfrac{7}{4} + \dfrac{27}{4} + 4 \cdot \dfrac{9}{4} + \dfrac{9}{4} = \dfrac{28}{4} + \dfrac{27}{4} + \dfrac{36}{4} + \dfrac{9}{4} = \dfrac{100}{4} = 25. \end{cases}$$

Answer: The concentration of acid in the first vessel is $\frac{7}{4}$ %, in the second one – $\frac{27}{4}$ %, in the third and fourth vessels – $\frac{9}{4}$ %.

Example 11.4 By mixing 30% and 60% acid solutions and adding 10 kg of pure water, a 36% acid solution was obtained. If, instead of 10 kg of water, we added 10 kg of a 50% solution of the same acid, we would get a 41% solution of the acid. How many kilograms of a 30% solution could be used to make a mixture?

Solution. Let the mass of the 30% acid solution be x kg, and the mass of the 60% acid solution be y kg. If 30% and 60% acid solutions are mixed and 10 kg of pure water is added, a 36% acid solution will be obtained:

$$0,3x+0,6y=0,36(x+y+10).$$

If instead of 10 kg of water, 10 kg of a 50% solution of the same acid was added, you would get a 41% solution of acid:

$$0,3x+0,6y+0,5\cdot 10=0,41(x+y+10).$$

Thus, we obtained the system of equations:

$$\begin{cases} 0,3x+0,6y=0,36(x+y+10); \\ 0,3x+0,6y+0,5\cdot 10=0,41(x+y+10), \end{cases}$$

$$\begin{cases} 0,3x+0,6y=0,36x+0,36y+3,6; \\ 0,3x+0,6y+5=0,41x+0,41y+4,1, \end{cases}$$

$$\begin{cases} 0,06x-0,24y=-3,6;\big| :0,06 \\ 0,11x-0,19y=0,9,\big| \cdot 10 \end{cases} \quad \begin{cases} x-4y=-60; \\ 11x-19y=90. \end{cases}$$

Let us solve the system by the inverse matrix method. To do this, we will find the inverse matrix for the matrix of the system:

$$A=\begin{pmatrix}1 & -4\\ 11 & -19\end{pmatrix},\quad \Delta A=\begin{vmatrix}1 & -4\\ 11 & -19\end{vmatrix}=-19+44=25,$$

$$A^{T}=\begin{pmatrix}1 & 11\\ -4 & -19\end{pmatrix},\quad A_{11}=-19,\quad A_{12}=4,\quad A_{21}=-11,\quad A_{22}=1,$$

$$A^{-1}=\frac{1}{25}\begin{pmatrix}-19 & 4\\ -11 & 1\end{pmatrix}.$$

Let us check the correctness of finding the inverse matrix:

$$A^{-1}\cdot A=\frac{1}{25}\begin{pmatrix}-19 & 4\\ -11 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & -4\\ 11 & -19\end{pmatrix}=\frac{1}{25}\begin{pmatrix}-19+44 & 76-76\\ -11+11 & 44-19\end{pmatrix}=$$

$$=\frac{1}{25}\begin{pmatrix}25 & 0\\ 0 & 25\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

The inverse matrix is found correctly. Find the unknowns of the system:

$$X=A^{-1}\cdot B=\frac{1}{25}\begin{pmatrix}-19 & 4\\ -11 & 1\end{pmatrix}\cdot\begin{pmatrix}-60\\ 90\end{pmatrix}=\frac{1}{25}\begin{pmatrix}1140+360\\ 660+90\end{pmatrix}=$$

$$=\frac{1}{25}\begin{pmatrix}1500\\ 750\end{pmatrix}=\begin{pmatrix}60\\ 30\end{pmatrix}.$$

Answer: 60 kg.

Tasks for revision

11.1. 5 g of magnesium chloride is obtained by treating 6.5 grams of a mixture of magnesium oxide and magnesium bromide with hydrochloric acid:

$$MgO + 2HCl \rightarrow MgCl_2 + H_2O,$$

$$MgBr_2 + 2HCl \rightarrow MgCl_2 + 2HBr.$$

Determine the composition of the mixture.

11.2. 16 g of a mixture of zinc, aluminum and copper were treated with an excess of hydrochloric acid solution. At the same time, 5.6 liters of gas were released and 5 g of the substance did not dissolve. Determine the mass proportion of metals in the mixture.

Note. Two metals react, and the third metal (copper) does not react. Therefore, a residue of 5 g is the mass of copper. The quantities of the other two metals, zinc and aluminum (note that their total mass is $16-5=11$ g) can be found using a system of equations.

12 Application of systems of linear algebraic equations in the calculation of electrical circuits

One method of analyzing an electrical circuit is the *method of loop currents*. It is based on the second Kirchhoff's law. Its main advantage is reducing the number of equations to $m-n+1$, where m is the number of branches, and n is the number of nodes in the chain. In practice, such a reduction greatly simplifies the calculation.

Contour current is a value that is the same in all branches of a given circuit. Usually in calculations they are denoted by double indices, for example, I_{11}, I_{22} etc.

The *actual current* in a particular branch is determined by the algebraic sum of the contour currents into which this branch enters. Finding the actual currents is the primary task of the loop current method.

Contour electromotive force (EMF) - is the sum of all the EMF included in this contour.

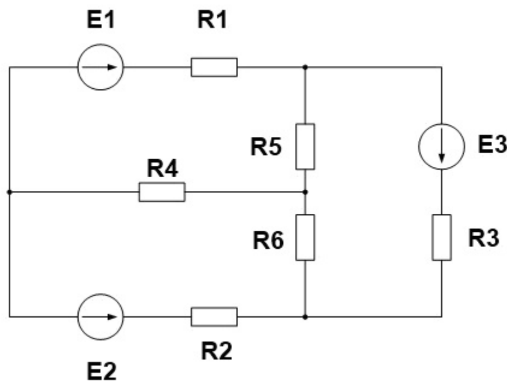
The *own resistance* of the contour is the sum of the resistances of all the branches that are included in it.

The *total resistance* of the contour is the branch resistance, adjacent to two contours.

The steps of the method of loop currents:

1. Selection of the actual currents direction.
2. The choice of independent contours and the directions of the contour currents in them.
3. Determination of the own and total resistances of the contours.
4. Making equations and finding the contour currents.
5. Finding actual currents.

Example 12.1

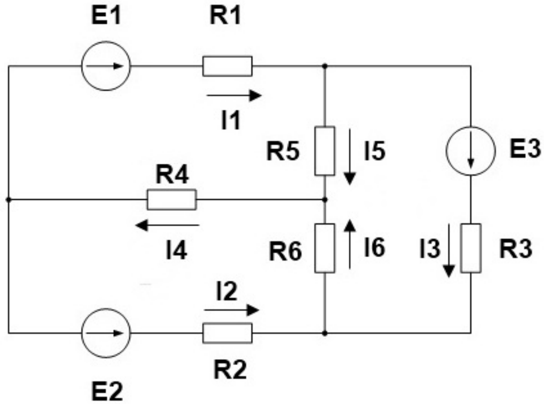


$E_1 = 80$ V, $E_2 = 50$ V, $E_3 = 60$ V, $R_1 = 10$ Ω, $R_2 = 15$ Ω, $R_3 = 30$ Ω, $R_4 = 25$ Ω, $R_5 = 30$ Ω, $R_6 = 35$ Ω,

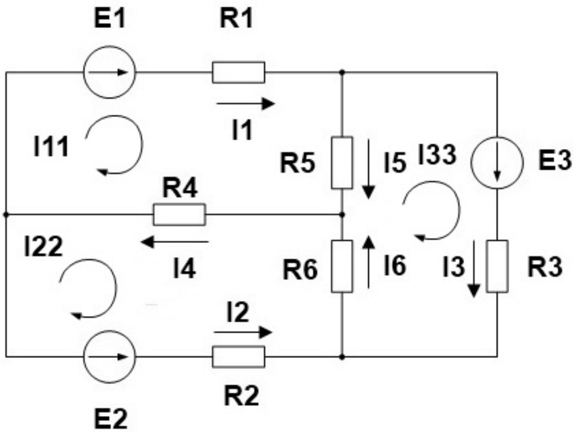
I_1, I_2, I_3, I_4, I_5, I_6 – ?

Solution.

1. Let us select the directions of actual currents $I_1 - I_6$.



2. We select three contours, and indicate the direction of the contour currents I_{11}, I_{22}, I_{33}. We select a clockwise direction.



3. Define the own resistances of the contours. To do this, we sum the resistances in each contour:

$$R_{11} = R_1 + R_4 + R_5 = 10 + 25 + 30 = 65 \ \Omega,$$

$$R_{22} = R_2 + R_4 + R_6 = 15 + 25 + 35 = 75 \ \Omega,$$

$$R_{33} = R_3 + R_5 + R_6 = 20 + 30 + 35 = 85\ \Omega.$$

Define the total resistances that belong to several contours at once, for example, the resistance R_4 belongs to contour 1 and contour 2. For convenience, we denote such resistances by the numbers of the contours to which they belong:

$$R_{12} = R_{21} = R_4 = 25\ \Omega,$$
$$R_{23} = R_{32} = R_6 = 35\ \Omega,$$
$$R_{31} = R_{13} = R_5 = 30\ \Omega.$$

4. Let us make the system of equations of contour currents. The left parts of the equations consist of the voltage drops in the circuit, and the right ones include the EMF of the sources of this circuit.

Since three contours are specified, the system will consist of three equations. For the first circuit, the equation will have the following form:

$$I_{11}R_{11} - I_{22}R_{21} - I_{33}R_{31} = E_1 .$$

The current I_{11} of the first contour is multiplied by the own resistance R_{11} of the same circuit, and then we subtract the current I_{22} multiplied by the total resistance of the first and second circuits R_{21}, and the current I_{33} multiplied by the total resistance of the first and third circuits R_{31}. This expression will be equal to the EMF E_1 of this contour. The value of the EMF is taken with a plus sign, since the direction of the circuit bypass (clockwise) coincides with the direction of the EMF. Otherwise, you need to take with a minus sign.

We perform the same actions with two other contours and as a result we get the system:

$$\begin{cases} I_{11}R_{11} - I_{22}R_{21} - I_{33}R_{31} = E_1; \\ I_{22}R_{22} - I_{11}R_{12} - I_{33}R_{32} = -E_2; \\ I_{33}R_{33} - I_{11}R_{13} - I_{22}R_{23} = E_3. \end{cases}$$

Let us substitute the already known values of resistances to the obtained system:

$$\begin{cases} 65I_{11} - 25I_{22} - 30I_{33} = 80; \\ 75I_{22} - 25I_{11} - 35I_{33} = -50; \\ 85I_{33} - 30I_{11} - 35I_{22} = 60, \end{cases} \quad \begin{cases} 13I_{11} - 5I_{22} - 6I_{33} = 16; \\ 15I_{22} - 5I_{11} - 7I_{33} = -10; \\ 17I_{33} - 6I_{11} - 7I_{22} = 12. \end{cases}$$

Solving the system (solve the system by yourself), we get:

$$\begin{cases} I_{11} = 2,726\,\text{A}; \\ I_{22} = 1,264\,\text{A}; \\ I_{33} = 2,189\,\text{A}. \end{cases}$$

5. Let us find the actual currents.

If the current flows only in one contour, then it is equal to the contour current (it is necessary to consider the direction of the bypass, for example, in our case the direction of the current I_2 does not coincide with the direction of the bypass, so we take it with a minus sign):

$$I_1 = I_{11} = 2,726 \text{ A}, \quad I_2 = -I_{22} = -1,264 \text{ A}, \quad I_3 = I_{33} = 2,189 \text{ A}.$$

The currents flowing through the total resistances are defined as the algebraic sum of the contour currents, taking into account the direction of the bypass. For example, a current I_4 flows through a resistor R_4, its direction coincides with the direction of the bypass of the first circuit and is opposite to the direction of the second circuit, therefore:

$$I_4 = I_{11} - I_{22} = 2,726 - 1,264 = 1,462 \text{ A}.$$

Similarly, for the rest currents we get:

$$I_5 = I_{11} - I_{33} = 0,537 \text{ A},$$

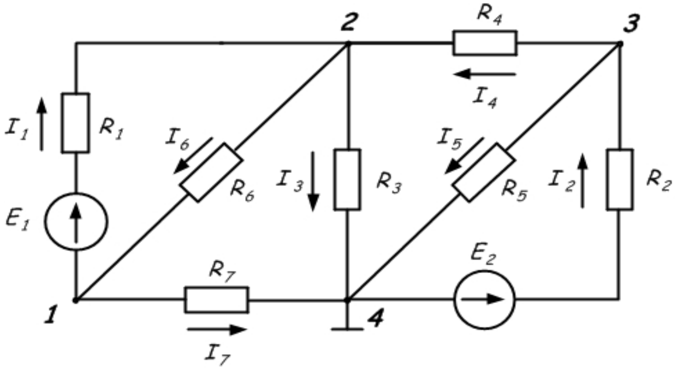
$$I_6 = I_{33} - I_{22} = 0,925 \text{ A}.$$

Answer: $I_1 = 2,726$ A, $I_2 = -1,264$ A, $I_3 = 2,189$ A, $I_4 = 1,462$ A, $I_5 = 0,537$ A, $I_6 = 0,925$ A.

The *node potential method* is one of the methods for analyzing an electrical circuit, which is advisable to use when the number of nodes in a circuit is less or equal to the number of independent contours. This method is based on the formulation of equations according to the first Kirchhoff's law. Herewith, the potential of one of the nodes of the chain is assumed to be zero, which allows reducing the number of equations to $n-1$.

Consider the work of the method by example.

Example 12.2



$R_1 = 25\ \Omega$, $R_2 = 22\ \Omega$, $R_3 = 42\ \Omega$, $R_4 = 35\ \Omega$, $R_5 = 51\ \Omega$, $R_6 = 10\ \Omega$, $R_7 = 47\ \Omega$, $E_1 = 50$ V, $E_2 = 100$ V,

I_1, I_2, I_3, I_4, I_5, I_6, I_7 – ?

Solution.

1. Let us take the fourth node as the base one and assume its potential equal to zero.

2. Make the equations according to the first Kirchhoff's law for 1, 2, 3 nodes (for node 4 we do not compile, since it is not required):

$$I_6 - I_7 - I_1 = 0,$$

$$I_1 + I_4 - I_6 - I_3 = 0,$$

$$I_2 - I_4 - I_5 = 0.$$

3. Using the generalized Ohm's law, we make equations for finding each of the currents (φ_i is the potential of the node from which the current goes, φ is the potential of the node into which the current enters, G_i is the conductivity of the i branch):

$$I_i = \frac{\varphi_i - \varphi + E_i}{R_i} = \left(\varphi_i - \varphi + E_i\right) G_i,$$

$$I_1 = \left(\varphi_1 - \varphi_2 + E_1\right) G_1, \quad I_2 = \left(0 - \varphi_3 + E_2\right) G_2,$$

$$I_3 = \left(\varphi_2 - 0 + 0\right) G_3, \quad I_4 = \left(\varphi_3 - \varphi_2 + 0\right) G_4,$$

$$I_5 = \left(\varphi_3 - 0 + 0\right) G_5, \quad I_6 = \left(\varphi_2 - \varphi_1 + 0\right) G_6, \quad I_7 = \left(\varphi_1 - 0 + 0\right) G_7.$$

4. Let us substitute the obtained expressions for currents in the equations from point 2:

$$\begin{cases} \varphi_1\left(G_1 + G_6 + G_7\right) + \varphi_2\left(-G_1 - G_6\right) = -E_1 G_1; \\ \varphi_1\left(-G_1 - G_6\right) + \varphi_2\left(G_1 + G_3 + G_4 + G_6\right) + \varphi_2\left(-G_4\right) = E_1 G_1; \\ \varphi_2\left(-G_4\right) + \varphi_3\left(G_2 + G_4 + G_5\right) = E_2 G_2. \end{cases}$$

The obtained system of equations was written for a chain consisting of 4 nodes, and for n nodes the following system holds:

$$\begin{cases} \varphi_1 G_{11} + \varphi_2 G_{12} + \ldots + \varphi_{n-1} G_{1,n-1} = \sum J_1 + \sum E_1 G_1; \\ \varphi_1 G_{21} + \varphi_2 G_{22} + \ldots + \varphi_{n-1} G_{2,n-1} = \sum J_2 + \sum E_2 G_2; \\ \qquad\qquad\qquad \ldots \\ \varphi_1 G_{n-1,1} + \varphi_2 G_{n-1,2} + \ldots + \varphi_{n-1} G_{n-1,n-1} = \sum J_{n-1} + \sum E_{n-1} G_{n-1}. \end{cases}$$

Conductances G_{11}, G_{22}, etc. are the sum of conductivities converging in a node (intrinsic conductivities) are always taken with a plus sign. Conductances G_{12}, G_{21}, etc. are the conductances of the branches connecting the nodes (total conductances) are always taken with a minus sign.

If the current source or EMF is directed to the node, then we take it with a plus sign, otherwise with a minus sign.

5. Solving the system of equations from point 4 (solve the system by yourself), we find unknown potentials in the nodes, and then we determine the currents:

$$\varphi_1 = 10{,}7 \text{ V}, \quad \varphi_2 = 26{,}6 \text{ V}, \quad \varphi_3 = 56{,}7 \text{ V},$$

$$I_1 = \left(\varphi_1 - \varphi_2 + E_1\right) G_1 = \left(10{,}7 - 26{,}6 + 50\right) \cdot 0{,}04 = 1{,}36 \text{ A},$$

$$I_2 = 1{,}97 \text{ A}, \quad I_3 = 0{,}63 \text{ A}, \quad I_4 = 0{,}86 \text{ A},$$

$$I_5 = 1{,}11 \text{ A}, \quad I_6 = 1{,}59 \text{ A}, \quad I_7 = 0{,}23 \text{ A}.$$

The correctness of the solution will be checked using the power balance:

$$I_1^2 R_1 + I_2^2 R_2 + I_3^2 R_3 + I_4^2 R_4 + I_5^2 R_5 + I_6^2 R_6 + I_7^2 R_7 = I_1 E_1 + I_2 E_2,$$

$$1{,}36^2 \cdot 25 + 1{,}97^2 \cdot 22 + 0{,}63^2 \cdot 42 + 0{,}86^2 \cdot 35 + 1{,}11^2 \cdot 51 +$$

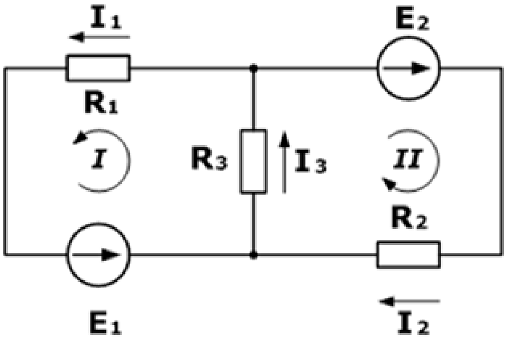
$$+1,59^2 \cdot 10 + 0,23^2 \cdot 47 = 1,36 \cdot 50 + 1,97 \cdot 100,$$

$$265,2 \text{ W} \approx 265,2 \text{ W}.$$

Answer: $I_1 = 1,36$ A, $I_2 = 1,97$ A, $I_3 = 0,63$ A, $I_4 = 0,86$ A, $I_5 = 1,11$ A, $I_6 = 1,59$ A, $I_7 = 0,23$ A.

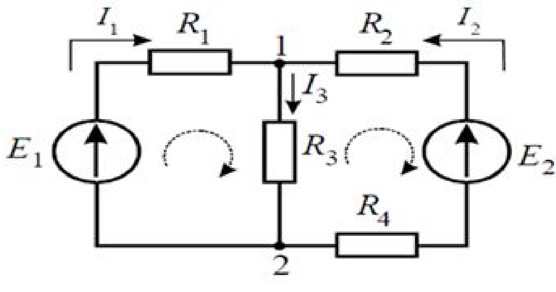
Tasks for revision

12.1



$E_1 = 75$ V,
$E_2 = 100$ V,
$R_1 = 100\ \Omega$,
$R_2 = 150\ \Omega$,
$R_3 = 150\ \Omega$,
$I_1,\ I_2,\ I_3 - ?$

12.2



$E_1 = 60$ V,
$E_2 = 450$ V,
$R_1 = 45\ \Omega$,
$R_2 = 15\ \Omega$,
$R_3 = 45\ \Omega$,
$R_4 = 75\ \Omega$,
$I_1,\ I_2,\ I_3 - ?$

ANSWERS

1.1. $\Delta = 2(ad - bc)$; 1.2. $\Delta = 59630$; 1.3. $\Delta = b^4 - 2b^3$;
2.1. $\det(AB) = \det(BA) = = \det A \cdot \det B = 4$;

2.2. $A^3 = 4a^3 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; 2.3. $A^{-1} = \frac{-1}{4}\begin{pmatrix} 1 & -3 & 1 \\ -3 & 1 & 1 \\ 1 & 1 & -3 \end{pmatrix}$;

3.1. $x_1 = 1, x_2 = 0$; 3.2. $x_1 = 1$, $x_2 = 2$, $x_3 = 1$, $x_4 = -1$;

4.1. $a = 2b$; 4.2. $X = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$; 4.3. $x_1 = -1$, $x_2 = 2$, $x_3 = 1$;

4.4. $x = 1$, $y = 2$, $z = 3$, $u = 4$, $v = 5$; 4.5. $rangA = 2$, $rang\tilde{A} = 3$, $rangA \neq rang\tilde{A}$, this system is incompatible;
5.1. $rangA = 3$; 5.2. $x_1 = 1$, $x_2 = 2$, $x_3 = 3$;
5.3. $x_1 = t,\ x_2 = t+1,\ x_2 = t+2,\ x_2 = t+3,\ t \in R$;

6.1. $A^{-1} = \frac{1}{18}\begin{pmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{pmatrix}$; 6.2. $A^{-1} = \frac{1}{40}\begin{pmatrix} -9 & 11 & 1 & 1 \\ 1 & -9 & 11 & 1 \\ 1 & 1 & -9 & 11 \\ 11 & 1 & 1 & -9 \end{pmatrix}$;

6.3. $A^{-1} = \frac{1}{4}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$; 6.4. $A^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$;

8.1. $x_1 = x_2 = x_3 = 0$; 8.2. $x_1 = -8t,\ x_2 = t,\ x_3 = t,\ t \in R$;

8.3. $x_1 = t,\ x_2 = \frac{-1}{8}t,\ x_3 = \frac{-1}{8}t,\ x_4 = 0,\ t \in R$;

9.1. $\lambda = a+b+c$, $\bar{x} = k\left(\vec{e_1}+\vec{e_2}+\vec{e_3}\right)$; 9.2. the eigenvalues are $\lambda_1 = 101$, $\lambda_2 = -99$, and their eigenvectors are $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$; 9.3. the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 6$ and their eigenvectors are $X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$;

10.1. $f(A) = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 15 & 2 & 0 \end{pmatrix}$; 10.2. $A = \begin{pmatrix} 28 & 15 & 16 \\ 19 & 36 & 15 \\ 30 & 19 & 28 \end{pmatrix}$;

10.3. . $A = \begin{pmatrix} 7a-1 & 0 & 0 \\ 7a & 7a-1 & 0 \\ 7a & 7a & 7a-1 \end{pmatrix}$;

11.1. $m(\mathrm{MgO}) = 0{,}9$ g, $m(\mathrm{MgBr_2}) = 5{,}6$ g;

11.2. 56,25% of zinc, 12,5% of aluminum, 31,25% of copper;

12.1. $I_1 = 0{,}143$ A, $I_2 = 0{,}262$ A, $I_3 = 0{,}405$ A;

12.2. $I_1 = 1{,}2$ A, $I_2 = 3{,}73$ A, $I_3 = 2{,}53$ A.

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ALPHABETICAL INDEX

APPENDICES

Appendix A

An example of the selection of principal unknowns and free unknowns

Example. Solve the system

$$\begin{cases} 3x_1 - 6x_2 + 9x_3 + 13x_4 = 9; \\ -x_1 + 2x_2 + x_3 + x_4 = -11; \\ x_1 - 2x_2 + 2x_3 + 3x_4 = 5. \end{cases}$$

If the system is uncertain, find a basic solution.

Solution. Consequently, we have a SLAE, which consists of three equations and four unknowns. Since the number of unknowns is greater than the number of equations, then such a system cannot have a unique solution, that is, this system is indefinite. Let's find a solution of the given system using Gaussian elimination method.

Write down the augmented matrix:

$$\tilde{A} = \left(\begin{array}{cccc|c} 3 & -6 & 9 & 13 & 9 \\ -1 & 2 & 1 & 1 & -11 \\ 1 & -2 & 2 & 3 & 5 \end{array} \right).$$

We need to get zeros at the first column. Swap the first and third rows:

$$\tilde{A} = \left(\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 5 \\ -1 & 2 & 1 & 1 & -11 \\ 3 & -6 & 9 & 13 & 9 \end{array} \right).$$

Add the elements of the first row to the elements of the second row. Then multiply the first row by (-3) and add to the third row:

$$\tilde{A} = \left(\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 5 \\ 0 & 0 & 3 & 4 & -6 \\ 0 & 0 & 3 & 4 & -6 \end{array} \right).$$

Let us get zeros in the third column. Multiply the second row by (-1) and add to the third one:

$$\tilde{A} = \left(\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 5 \\ 0 & 0 & 3 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

The straight course of the Gaussian elimination method is complete. We reduced the augmented matrix to the echelon form. The number of non-zero lines of the augmented matrix $\tilde{A}$ and of the coefficient matrix A is equal to two but it is less than a number of unknowns:

$$rang\ \tilde{A} = rang\ A = 2 = r < n,\ \ n = 4\ .$$

Consequently, the given system is indefinite, that is, it has an infinite set of solutions. Find these solutions. First of all we should select the principal unknowns. Their number should be equal r, in our example it is $r = 2$. We select as the principal unknowns the ones which are located at the first places of the non-zero rows of the resulting stepwise matrix, that is, on the "steps". There are first and third columns. The first column corresponds to the unknown x_1, and the third column corresponds to the unknown x_3. We can take the variables x_1, x_3 as the principal unknowns, because the minor consists of the first and third columns is not equal to zero:

$$\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 1\cdot 3 - 0\cdot 2 = 3 \neq 0 \,,$$

it is the principal.

Unknowns x_2 and x_4 are free unknowns. Let's express the principal unknowns x_1 and x_3 through free x_2 and x_4. Write the obtained matrix:

$$\tilde{A} = \left(\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 5 \\ 0 & 0 & 3 & 4 & -6 \end{array}\right)$$

Rewrite it in the system form:

$$\begin{cases} x_1 - 2x_2 + 2x_3 + 3x_4 = 5; \\ 3x_3 + 4x_4 = -6. \end{cases}$$

Present expression for unknown x_3 from the second equation and substitute it into the first row:

$$\begin{cases} x_1 - 2x_2 + 2\cdot\frac{1}{3}(-4x_4 - 6) + 3x_4 = 5; \\ x_3 = \frac{1}{3}(-4x_4 - 6). \end{cases}$$

The general solution of the system can be write as

$$\begin{cases} x_1 = 2x_2 - \frac{1}{3}x_4 + 9; \\ x_2 \in R; \\ x_3 = -\frac{4}{3}x_4 - 2; \\ x_4 \in R. \end{cases}$$

The partial solution of the system can be found due to putting value of unknowns $x_2 = 0,\ x_4 = 0$ in the general solution and it is:

$$\begin{cases} x_1 = 9; \\ x_2 = 0; \\ x_3 = -2; \\ x_4 = 0. \end{cases}$$

Answer: $x_1 = 9$, $x_2 = 0$, $x_3 = -\frac{4}{3}x_4 - 2$, $x_4 = 0$.

Appendix B

Linear spaces

A linear (vector) space L over a field K is a set with two binary operations:

1) $L \times L \to L$ is usually designated as an addition: $(l_1, l_2) \mapsto l_1 + l_2$,

2) $K \times L \to L$ is usually denoted as a multiplication: $(a, l) \mapsto al$.

These operations satisfy the following axioms:

a) adding elements of L, or vectors, turns L into a commutative (abelian) group. Its zero element is usually denoted as 0; an element, inverse to l, is denoted as $-l$;

b) multiplication of vectors by the elements of a field K, or scalars, is unitary, that is $1 \cdot l = 1$, for all l, and associative, that is $a(bl) = (ab)l$, for all $a, b \in K$; $l \in L$;

c) adding and subtracting are connected by the distributivity laws, i.e.

$$a(l_1 + l_2) = al_1 + al_2,\ (a_1 + a_2)l = a_1 l + a_2 l$$

for all $a_1, a_2, a \in K$; $l, l_1, l_2 \in L$.

The expression $\sum_{i=1}^{n} a_i l_i$ is called *a linear combination* of vectors $l_1, ..., l_n$; the scalars a_i are the coefficients of this linear combination.

Zero-dimensional space is an abelian group $L = \{0\}$, which consists of only zero element. The single possible law of multiplication on scalars is $a \cdot 0 = 0$ for all $a \in K$.

The main field K can be considered as *a one-dimensional coordinate space* with $L = K$, adding is the addition in K, multiplying by a scalar is the multiplication in

K. For example, the field of complex numbers is a linear space over the field of real numbers, which is a linear space over the field of rational numbers.

Let $L = \mathrm{K}^n = \mathrm{K} \times ... \times \mathrm{K}$ (Cartesian product $n \geq 1$ of multipliers), then $L = \mathrm{K}^n = \mathrm{K} \times ... \times \mathrm{K}$ is *n-dimensional coordinate space.* Items of L are written as rows $(a_1, ..., a_n), a_i \in \mathrm{K}$ or columns $\begin{pmatrix} a_1 \\ ... \\ a_n \end{pmatrix}$. One-dimensional spaces over a field K are called straight lines; two-dimensional are planes.

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